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GROUP PROPERTIES OF THE FINITE-DIMENSIONAL LINEARIZATION OPERATOR OF A DYNAMICAL SYSTEM IN THE MODEL OF ORBITAL TETHER SYSTEM MOTION

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Abstract

The article considers an autonomous linear dynamical system represented by ordinary differential equations that define the motion of an orbital tether system. The object of the study are group properties of the spectrum of a finite-dimensional linearization operator, which smoothly depends on the control parameter k, in the case of general position – without degeneracy. Conditions are defined under which there emerges a group of symplectomorphisms generating in a linear system the first integral in the form of a nondegenerate quadratic form – Hamilton functions. The existence of a symplectic structure and a quadratic invariant in a dynamical system allows to reduce it on the basis of the variational principle to a divergent Hamiltonian form of the equations of motion with the linearization operator represented in a certain "canonical" form. The distinguished category of systems with a single invariant allows both to construct a Lie algebra of the corresponding Lie group and to move on to the study of stability and gyroscopic stabilization in a mechanical system.

Keywords: group approach, symplectic structure, quadratic invariant, linear Hamiltonian system, Lie algebra.

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DIMENSÕES FINITAS DE UM SISTEMA DINÂMICO NO MODELO DE MOVIMENTO DO SISTEMA ORBITAL TETHER

Resumo

O artigo considera um sistema dinâmico linear autônomo representado por equações diferenciais ordinárias que definem o movimento de um sistema de corda orbital. O objeto do estudo são propriedades de grupo do espectro de um operador de linearização de dimensão finita, que depende suavemente do parâmetro de controle k, no caso de posição geral – sem degenerescência. São definidas condições sob as quais emerge um grupo de simplectomorfismos gerando em um sistema linear a primeira integral na forma de uma forma quadrática não degenerada – funções de Hamilton. A existência de uma estrutura simplética e de um invariante quadrático em um sistema dinâmico permite reduzi-lo com base no princípio variacional a uma forma hamiltoniana divergente das equações de movimento com o operador de linearização representado em uma certa forma "canônica". A distinta categoria de sistemas com um único invariante permite construir uma álgebra de Lie do grupo de Lie correspondente e passar para o estudo da estabilidade e estabilização giroscópica em um sistema mecânico.

Palavras-chave: abordagem de grupo, estrutura simplética, invariante quadrático, sistema hamiltoniano linear, álgebra de Lie.

PROPIEDADES DE GRUPO DEL OPERADOR DE LINEALIZACIÓN DE DIMENSIONES FINITAS DE UN SISTEMA DINÁMICO EN EL MODELO DE MOVIMIENTO DEL SISTEMA DE ANCLAJE ORBITAL

Resumen

El artículo considera un sistema dinámico lineal autónomo representado por ecuaciones diferenciales ordinarias que definen el movimiento de un sistema de amarre orbital. El objeto de estudio son las propiedades de grupo del espectro de un operador de linealización de dimensión finita, que depende suavemente del parámetro de control k, en el caso de posición general, sin degeneración. Se definen condiciones bajo las cuales emerge un grupo de simplectomorfismos que generan en un sistema lineal la primera integral en forma de forma cuadrática no degenerada – funciones de Hamilton. La existencia de una estructura simpléctica y una invariante cuadrática en un sistema dinámico permite reducirlo sobre la base del principio variacional a una forma hamiltoniana divergente de las ecuaciones de movimiento con el operador de linealización representado en cierta forma "canónica". La distinguida categoría de sistemas con un solo invariante permite tanto construir un álgebra de Lie del grupo de Lie correspondiente como pasar al estudio de la estabilidad y estabilización giroscópica en un sistema mecánico.

Palabras clave: enfoque grupal, estructura simpléctica, invariante cuadrática, sistema hamiltoniano lineal, álgebra de Lie.

1. INTRODUCTION

One of the topical issues in the study of dynamical systems represented by differential equations is the application of qualitative methods and approaches, that is, the analysis of solutions to a differential equation by its analytic part given on some manifold. There there arises the problem of studying the properties of integral curves, their topology, behavior in the vicinity of special points: the issues of stability, gyroscopic stabilization, "roughness" (structural stability), etc. (Neishtadt, Treschev, 2021).

One of the qualitative methods employed in the study of this issue is the group approach.

A group (semigroup), being one of the simplest algebraic constructions with a given binary operation of associative multiplication, understood as a mapping, allows us to reveal the most fundamental (significant) properties of the object under study.

If $\{G\}$ is a manifold of elements, then G is a typical representative of the manifold, and a binary operation is given.

$\{G\} \otimes \{G\} \to \{G\}, then \ (\{G\};\otimes)$ is a groupoid.

A groupoid with an associative binary operation, such as matrix multiplication, forms a semigroup. The existence of an identity element, that is, the existence of a left and a right identity element for the same semigroup, turns the semigroup into a monoid: $E_L \cdot G = G \cdot E_R = EG = G$. The existence of an inverse element for $G: G^{-1} \cdot G = G \cdot G^{-1} = E, G^{-1} \in \{G\}$ generates a group from a monoid.

2. METHODS

Consider a dynamic system given by ordinary differential equations describing the motion of an orbital tether system in circular orbits (Beletsky, Levin, 1993; Yu et al., 2018; Dadashov, Lapir, 2020), which when parameterized by $t (t \geq 0)$ takes the form:

$$\begin{cases} \dot{\Omega}_{orb} = \frac{3}{2} sin2\varepsilon - 2k(1 + \Omega_{orb}) \\ \dot{\varepsilon} = \Omega_{orb} \end{cases} \quad (1)$$

$k - control\, parameter\ (k > 0)$

Let us introduce for (1) the notation $\sum_t -$ the dynamic system that defines the transformation of space $M^2 \subset R^2 \to M^2 (manifold)$ with a group property:

$$\begin{cases} \sum_{t+s} = \sum_t \circ \sum_s \\ \sum_{-t} = (\sum_t)^{-1} \end{cases} \quad (2)$$

Linearizing (1) by powers of $x_1 = \Omega_{orb} - \Omega_0$and $x_2 = \varepsilon - \varepsilon_0$ in the vicinity of the equilibrium position (special trajectories are fixed points), we obtain:

$$\left\|\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix}\right\| = \begin{pmatrix} -2k & 3cos2\varepsilon_0 \\ 1 & 0 \end{pmatrix}\Big|_{(\Omega_0;\varepsilon_0)} \cdot \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\| \text{ or } \dot{x} = \begin{pmatrix} -2k & \theta \\ 1 & 0 \end{pmatrix} \cdot x \Leftrightarrow \dot{x} = A(k;\theta) \cdot x \quad (3)$$

under initial conditions (fixed points):

$$\begin{cases} \Omega_0 \equiv 0 \\ sin2\varepsilon_0 = \frac{4k}{3} \\ cos2\varepsilon_0 \equiv \frac{\theta}{3} \end{cases} \Leftrightarrow \begin{cases} \left(\frac{\theta}{3}\right)^2 + \left(\frac{k}{\frac{3}{4}}\right)^2 \equiv 1 \\ \Omega_0 \equiv 0 \end{cases} \quad (4),$$

while $A(k;\theta)$ is a linear operator in the canonical Frobenius form with the characteristic polynomial

$$\det(A(k;\theta) - \lambda E) = 0 \Leftrightarrow \lambda^2 + 2k\lambda - \theta = 0 \quad (5)$$

Let us consider the manifold of matrices

$GL_2(R) = \{Mat_{2x2}(R): det\,(Mat_{2x2}(R)) \neq 0\} -$ general linear group of the rank 2; $A(k;\theta) \subset \{Mat_{2x2}(R)\}$ with $\theta \neq 0 \left(k \neq \frac{3}{4}\right)$. $GL_2(R)$ forms a group with respect to the matrix multiplication operation. A general linear group preserves the basis of the linear space on R^2 (with the given operations of adding linear elements and their multiplication by a number from R). A degenerate case of $\theta \neq 0 \left(k \neq \frac{3}{4}\right)$ specifies membership in a semigroup with the right identical (neutral) element $E_R = \begin{pmatrix} 1 & 0 \\ r & 0 \end{pmatrix} : \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right\}$, where $a; b; r \in R,$ is an example of an algebraic structure in which there is no left $E_L = \begin{pmatrix} 1 & l \\ 0 & 0 \end{pmatrix}$ identical element, but there does exist an infinite manifold of right identical elements.

The group $GL_2(R)$contains in itself as a subgroup a manifold of matrices above the plane R with $|detA(k;\theta)| = 1 -$ a group of unimodular matrices with the determinant ± 1, under the condition $\theta = \mp 1$.

To the condition $\theta = -1$ corresponds a special linear group of rank 2:

$SL_2(R) = \{Mat_{2x2}(R): det\,(Mat_{2x2}(R)) = 1\}$, which is a subgroup in $GL_2(R)$ preserving a 2-linear (bilinear) antisymmetric (cosymmetric) form.

The condition $\theta = -1$ defines a linear transformation of the symplectic space $\Omega: R^2 \to R^2$ is for which it is necessary and sufficient to preserve the bilinear cosymmetric form:

$\Omega^T \cdot I \cdot \Omega = I$, where $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is symplectic unit; $I^2 = -E$;

$\Omega \equiv A(k;\theta)|_{\substack{k=\frac{\sqrt{2}}{2} \\ \theta=-1}} = A\left(\frac{\sqrt{2}}{2};-1\right)$.

The Frobenius characteristic polynomial transforms into a return polynomial at $\theta = -1$:

$$\lambda^2 + 2k \cdot \lambda + 1 = 0, \text{ thus } \det(\Omega - \lambda E) = \lambda^2 \cdot \det\left(\Omega - \frac{1}{\lambda} E\right) \qquad (6)$$

Since λ and $\frac{1}{\lambda}$ as the eigenvalues of the symplectic transformation ($\lambda = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot i$) lie on an identical circle ($|\lambda| = 1$), and, accordingly, transformation Ω is highly stable, then each $\tilde{\Omega}$ that is close enough to Ω is stable (elements of the matrices Ω *and* $\tilde{\Omega}$ differ less than by the rather small value ε).

Considering that λ and $\bar{\lambda} = \frac{1}{\lambda}$ are of multiplicity 1, their corresponding two-dimensional invariant plane is nonzero. Thus, $\Omega = A(\frac{\sqrt{2}}{2};-1)$ belongs to the symplectic group $Sp_2(R) = SL_2(R) \Leftrightarrow \theta = -1$.

In the general case, $Sp_2(R) \subset SL_2(R)$, while $Sp_2(R) = \{Mat_{2x2}(R): det\, det\left(Mat_{2x2}(R)\right) = 1; Mat_{2x2}(R) \cdot I \cdot Mat_{2x2}(R) = I\}$ is a group of symplectomorphisms, that is, diffeomorphisms between regions in R^2 that retain a bilinear cosymmetric form when the inverse image is mapped:

$$\Omega^T \cdot I \cdot \Omega = I \Leftrightarrow I^{-1} \cdot \Omega^T \cdot I = \Omega^{-1} \ (7)$$

Ω^T and Ω^{-1} are are similar, hence for Ω; Ω^T *and* Ω^{-1} the characteristic polynomial is invariant.

To the bifurcation value $k = \frac{\sqrt{2}}{2}$ correspond two values of the parameter $\theta = \pm 1$, and $\theta = -1$ is a symplectic case.

Let $\dot{x} = A(k;\theta) \cdot x$ be a linearized system (3), where $A = \begin{pmatrix} -2k & \theta \\ 1 & 0 \end{pmatrix}$ is the matrix of operator $A(k;\theta)$, then if $\Omega = \begin{pmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}$ is the symplectic matrix for the operator

A$(k;\theta)$, then there exists a quadratic invariant $\Psi(x) \equiv \frac{1}{2}(\Psi(k;\theta)\cdot x;x)$, such that $\Psi^*(k;\theta) \equiv \Psi(k;\theta) = \Omega \cdot A(k;\theta)$ is a self-adjoint operator set by the symmetric matrix $\Psi(k,\theta)$:

$\Psi(k;\theta) = \Omega \cdot A(k;\theta) = \begin{pmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2k & \theta \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2}k - 1 & -\sqrt{2}\theta \\ -2k & \theta \end{pmatrix}$; the condition $\Psi^*(k;\theta) = \Psi(k;\theta)$ (8)

leads to the following ratio:

$-\sqrt{2}\theta = -2k \Rightarrow \theta = \sqrt{2}k$ and taking into account (4) we obtain $k = \frac{\sqrt{2}}{2}$; $\theta=1$, which means that $A = \begin{pmatrix} -\sqrt{2} & 1 \\ 1 & 0 \end{pmatrix}$ is a self-adjoint operator (symmetric matrix): $A^* = A$, which with $k = \frac{\sqrt{2}}{2}$; $\theta=1$ sets the self-adjoint operator $\Psi(k,\theta)$ with matrix $\Psi(\frac{\sqrt{2}}{2};1) = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$ the quadratic form $\Psi(x)$.

Note that $\frac{d\Psi(x)}{dt} = \frac{1}{2}((\Psi(k,\theta)\cdot A + A^* \cdot \Psi(k,\theta))\cdot x;x) \le 0 \quad \forall x \in M^2 \subset R^2$, that is, $\frac{d\Psi(x)}{dt}$ does not increase along the equations of motion. Indeed (Dadashov, Lapir, 2020; Kozlov, 2013), $\Psi(k,\theta)\cdot A(k,\theta) + A^*(k,\theta)\cdot\Psi(k,\theta)$ with $k = \frac{\sqrt{2}}{2}$; $\theta=1$ takes the form:

$$\begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} -\sqrt{2} & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\sqrt{2} & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} & 1 \\ 3 & -\sqrt{2} \end{pmatrix} + \begin{pmatrix} -2\sqrt{2} & 3 \\ 1 & -\sqrt{2} \end{pmatrix} = \begin{pmatrix} -4\sqrt{2} & 4 \\ 4 & -2\sqrt{2} \end{pmatrix},$$

therefore,

$$\frac{d\Psi(x)}{dt} = \frac{1}{2}\left(\begin{pmatrix} -4\sqrt{2} & 4 \\ 4 & -2\sqrt{2} \end{pmatrix} \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\|; \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\|\right) = \frac{1}{2}\left(-4\sqrt{2}x_1^2 + 8x_1x_2 - 2\sqrt{2}x_2^2\right) == -2\sqrt{2}x_1^2 + 4x_1x_2 - \sqrt{2}x_2^2 = -\sqrt{2}\left(2x_1^2 - 2\sqrt{2}x_1x_2 + x_2^2\right) = -\sqrt{2}\left(\sqrt{2}x_1 - x_2\right)^2 \le 0$$

and $\Psi(x)$ is a quadratic invariant.

The existence of a quadratic invariant $\Psi(x)$entails the appearance of an integer spectrum of quadratic invariants $\Psi_m(x)$ (Kozlov, 2019) with a symmetric matrix $\Psi_m(k;\theta)$:

$(A^*)^m \cdot \Psi(k,\theta) \cdot (A)^m = \Psi_m(k,\theta)$ (9)

Meanwhile, is the spectrum of operator $A(k;\theta)$ is simple, then $\Psi(x); \Psi_1(x); \Psi_2(x)$ … are functionally independent (given that det(A-E)$\neq$ 0), that is, with m=1; for example:

$A^*(k,\theta)\cdot\Psi(k,\theta)\cdot A(k,\theta) = A^*\left(\frac{\sqrt{2}}{2};1\right)\cdot\Psi\left(\frac{\sqrt{2}}{2};1\right)\cdot A\left(\frac{\sqrt{2}}{2};1\right) = \begin{pmatrix}-\sqrt{2} & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}1 & -\sqrt{2}\\ -\sqrt{2} & 1\end{pmatrix}\cdot\begin{pmatrix}-\sqrt{2} & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}7 & -2\sqrt{2}\\ -2\sqrt{2} & 1\end{pmatrix} = \Psi_1\left(\frac{\sqrt{2}}{2};1\right); det\Psi(\frac{\sqrt{2}}{2};1) == det\Psi_1\left(\frac{\sqrt{2}}{2};1\right) = -1$, so (8) preserves the measure (volume) and orientation of space.

In the general case, there are $\frac{n}{2}$ independent (functionally) (Zheglov et al., 2019; Kozlov, 2018) quadratic invariants in R^n, i.e. with $n = 2(R^2)$: $\Psi(x)$ or $\Psi_1(x)$, for instance: $\Psi(x); \Psi_1(x); \Psi_2(x)$ … will no longer be independent.

The condition $x_1 \equiv x_2 \equiv 0$ and $x_1 \equiv kx_2$ with $k = \frac{\sqrt{2}}{2}$ determine $\frac{d\Psi(x)}{dt} \equiv 0 \Leftrightarrow$

$\Leftrightarrow \Psi(k,\theta)\cdot\tilde{A} + \tilde{A}^*\cdot\Psi(k,\theta) = 0$ (10)

with $k = \frac{\sqrt{2}}{2}$; θ=1, that is, a Lie algebra $\{g_{\psi(k,\theta)}\}$ of the group $\{G_{\psi(k,\theta)}\}$ is given:

$$\tilde{A} \in \{g_{\psi(k,\theta)}\}, if \ \exists \text{ is a smooth curve of } g(t) \in \{G_{\psi(k,\theta)}\}:$$
$$\left.\frac{dg}{dt}\right|_{t=0} = \tilde{A}, \text{meanwhile}, g(t) \in \{G_{\psi(k,\theta)}\} \Leftrightarrow$$
$$\Leftrightarrow (g(t))^T\cdot\Psi(k,\theta)\cdot(g(t)) = \Psi(k,\theta), \text{ where } k = \frac{\sqrt{2}}{2}; \theta=1;$$
$$\left.exp\{\tilde{A}t\}\right|_{t=1} = \left.exp\{\tilde{A}\}\right|_{t=1} \in \{G_{\psi(k,\theta)}\}$$

Relationship (10) is satisfied by the manifold of matrices of the form:

$\tilde{A}(\gamma) = \gamma\cdot\begin{pmatrix}\sqrt{2} & -1\\ 1 & -\sqrt{2}\end{pmatrix}, \gamma \in R.$ (11)

Similarly, consider the Lie algebra $\{g_\Omega\}$:

$\{g_\Omega\}: \Omega\cdot\left(A\left(\frac{\sqrt{2}}{2};1\right)\cdot\Phi\right) + \left(A\left(\frac{\sqrt{2}}{2};1\right)\cdot\Phi\right)^*\cdot\Omega = 0$, of the group

$\{G_\Omega\}: (g(t))^T\cdot\Omega\cdot(g(t)) = \Omega$, while $\{g_\Omega\}$ is isomorphic to the symplectic Lie algebra $sp(2;R)$;then $\Phi\cdot\Psi\left(\frac{\sqrt{2}}{2};1\right) - \Psi\left(\frac{\sqrt{2}}{2};1\right)\cdot\Phi = 0$

Let $\Phi \equiv \Phi^*$ be the is a self-adjoint operator given by the matrix $\Phi \equiv \Phi^T$, then the commutator $[\Psi;\Phi] = [\Phi;\Psi] = 0; [\Psi_1;\Phi_1] = [\Phi_1;\Psi_1] = 0$ and so on for each pair of Ψ_m and Φ_m, therefore $\{\Phi;\Phi_1;\dots;\Phi_m\} = \xi(g_{\psi_m})$ is the center of Lie algebra $g_{\Psi_m} = \{\Psi;\Psi_1;\dots;\Psi_m;\Phi;\Phi_1;\dots;\Phi_m\}$ of a Cartan subalgebra consisting of self-adjoint operators defining the quadratic invariants $\Psi_m(x) = \frac{1}{2}(\Psi_m\cdot x;x)$ and $\Phi_m(x) = \frac{1}{2}(\Phi_m\cdot x;x)$, respectively.

Examination of the structure of the quadratic form $\Psi(x)$makes it possible to determine the degree of instability of the system (3) with $k = \frac{\sqrt{2}}{2}$; θ=1. Let u be the degree of instability, that is, the number of eigenvalues of the operator $A(k;\theta)$ in the right half-plane of the complex plane (Fig. 1).

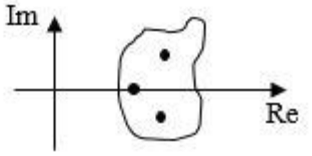

Figure 1. Complex plane

Let i^{-} be the negative index of inertia of the quadratic form $\Psi(x)$, accordingly, $i^{+}$is the positive index; then $u \equiv i^{-}(mod2)$, $i^{-} + i^{+} = n = 2$ (in our case) (Kozlov, 1992)

$$\Psi(x) = \frac{1}{2}\left(\begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \cdot \left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\| ; \left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\| \right) = x_1^2 + x_2^2 - 2\sqrt{2}x_1 \cdot x_2,$$

which is reduced by a nondegenerate transformation to the form: $\Psi(z) = -z_1^2 + z_2^2$; $i^{-} = 1$ and the system (3) is unstable.

Here, in the general case, the presence of a positively defined quadratic invariant $\Psi(x)$ defines the stability of the system (3).

Since the negative index of inertia $i^{-} = 1$, then the spectrum $\sigma\left(A(\frac{\sqrt{2}}{2}; 1)\right)$ consists of two real eigenvalues λ_1 *and* λ_2: $\lambda_1 = \frac{-(1+\sqrt{3})}{\sqrt{2}}$; $\lambda_2 = \frac{\sqrt{3}-1}{\sqrt{2}}$, which is located in the right half-plane of the complex plane (Fig. 1)

$\lambda_1 \cdot \lambda_2 < 0; det\left(A\left(\frac{\sqrt{2}}{2}; 1\right)\right) < 0$; unstable saddle.

The existence for the operator $A(k; \theta)$ with $k = \frac{\sqrt{2}}{2}$ of a bifurcation point on $\theta(\theta = \pm 1)$saddle-focus may indicate a bifurcation of the formation of the saddle-focus separatrix loop.

Let point P be the saddle-focus in R^3 with a one-dimensional unstable subspace of the operator A: $\lambda = \frac{-1\pm\sqrt{3}}{\sqrt{2}}$ and a two-dimensional stable manifold $Re\lambda_{1,2} = \frac{-1}{\sqrt{2}}$ of operator $\Omega = A(\frac{\sqrt{2}}{2}; -1)$.

Given that the first and second saddle values, respectively, are:

$$\sigma_1(k) = \sigma_1\left(\frac{\sqrt{2}}{2}\right) = Re\lambda_{1,2} + \lambda < 0$$
$$\sigma_2(k) = \sigma_2\left(\frac{\sqrt{2}}{2}\right) = 2Re\lambda_{1,2} + \lambda < 0;$$

Then if $k = \frac{\sqrt{2}}{2}$ (bifurcation point), the system has a homoclinic trajectory Υ, which, leaving the point P, returns to the same point at $t \to \infty$ (rather long time).

Condition $\sigma_1 < 0 \wedge \sigma_2 < 0$suggests that there is no there is no nontrivial hyperbolic manifold (a countable set of periodic attractors from the phase trajectories of the system), and hence the loop of the homoclinic curve Υ is destroyed, which, in turn, leads either to the birth of a stable cycle (I), or not (II) (Fig. 2).

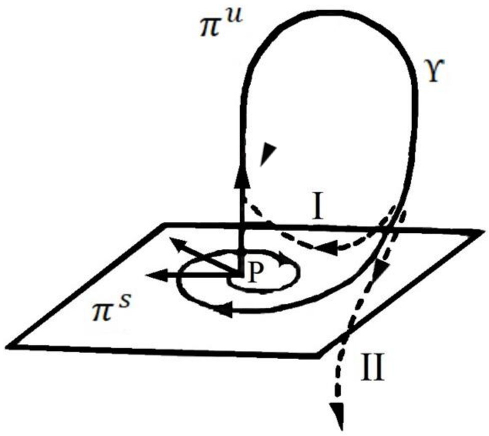

Figure 2. Separatrix loop at point P (saddle-focus)

Consider that the cone $\left\{x \in R^2 \backslash \{0\}: \frac{d\Psi(x)}{dt} \equiv 0\right\}$ does not contain closed trajectories, then when the degree of instability $u = i^{-=1}$,case II is realized (Fig. 2) – there are no limit cycles (I).

For a more detailed study of nonlocal bifurcations (saddle-focus, for example) in the vicinity of the point itself, methods of nonlinear analysis are required, and these studies were not conducted in this paper; however, let us note that condition (8) is equivalent to the problem on a conditional extremum for a quadratic form of the form:

$$x^T \cdot A(k;\theta) \cdot x \to extr\ on\ the\ compact\ (sphere)\ x^T \cdot x = 1: \tag{12}$$

$$\mathrm{f} = x^T \cdot A(k,\theta) \cdot \mathrm{x} + \lambda(1 - x^T \cdot \mathrm{x});$$

f – Lagrange function, $\lambda -$ Lagrange multiplier.

Stationarity condition:

$$\frac{\partial \mathrm{f}}{\partial x} = x^T \cdot A(k,\theta) + x^T A^*(\mathrm{k},\theta) - 2\lambda x^T \equiv 0,\ while\ \mathrm{A}(\mathrm{k},\theta) \equiv A^*(\mathrm{k},\theta)\Big|_{k=\frac{\sqrt{2}}{2}}\ \theta = 1$$

$$-\ \text{the condition of self} - \text{adjacency}$$

$\frac{\partial f}{\partial x} = 2x^T \cdot A(\frac{\sqrt{2}}{2},1)$ - $2\lambda x^T \equiv 0$ or $A(\frac{\sqrt{2}}{2},1) \cdot x - \lambda x \equiv 0$, that is, f(x)→extr, then x- is the eigenvector of the operator $A(\frac{\sqrt{2}}{2},1)$, corresponding to the eigenvalue of λ.

Sufficient conditions of extremum for f are given by Hessian $det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \equiv 0$ – no unconditional extremum, and Hessian $det\left(\frac{\partial^2 (x^T A x)}{\partial x_i \partial x_j}\right) < 0$ – no conditional extremum.

Solution of the dynamic system (3) $\dot{x} = A(k,\theta) \cdot x$, adjoint to the invariant $\Psi(x)$ is equivalent to the solution of the Hamiltonian system with Hamilton function H: $k = \frac{\sqrt{2}}{2}$; θ=1,

$\left\|\begin{matrix}\dot{x}_1\\ \dot{x}_2\end{matrix}\right\| = \begin{pmatrix}-\sqrt{2} & 1\\ 1 & 0\end{pmatrix}\left\|\begin{matrix}x_1\\ x_2\end{matrix}\right\| \Leftrightarrow \left\|\begin{matrix}\dot{x}_1\\ \dot{x}_2\end{matrix}\right\| = \left(\Omega^{-1} \cdot \Psi\left(\frac{\sqrt{2}}{2};1\right)\right) \cdot \left\|\begin{matrix}x_1\\ x_2\end{matrix}\right\|$ or $\dot{x} = \Omega^{-1} \cdot \frac{\partial \Psi(x)}{\partial x}$ (13), in other words, $\dot{x} = A(\frac{\sqrt{2}}{2};1) \cdot x$ is the Hamiltonian system with Hamilton function $H = \Psi(x)$.

Indeed, $\Omega = \begin{pmatrix}-\sqrt{2} & -1\\ 1 & 0\end{pmatrix}$; $\Omega^{-1} = \begin{pmatrix}0 & 1\\ -1 & -\sqrt{2}\end{pmatrix}$; $\Omega^{-1} \cdot \Psi\left(\frac{\sqrt{2}}{2};1\right) = \begin{pmatrix}0 & 1\\ -1 & -\sqrt{2}\end{pmatrix} \cdot \begin{pmatrix}1 & -\sqrt{2}\\ -\sqrt{2} & 1\end{pmatrix} = \begin{pmatrix}-\sqrt{2} & 1\\ 1 & 0\end{pmatrix} = A(\frac{\sqrt{2}}{2};1)$ – is the symmetric matrix A of the self-adjoint operator $A\left(\frac{\sqrt{2}}{2};1\right)$.

The Hamilton equations (13) generate a one-parameter group of phase space transformations in itself: $g^t_{x_0}: R^2 \to R^2 - phaseflux$.

As $g^t_{x_0}: x_0 \to x(t, x_0)$, then the Jacobian $J = det\left(\frac{\partial x}{\partial x_0}\right) \neq 0$ and if $(\frac{\partial x}{\partial x_0})^T \cdot I \cdot \left(\frac{\partial x}{\partial x_0}\right) = I$, then the Jacobi matrix J defines a symplectic structure.

For example, Ω : $\left(\frac{\partial x}{\partial x_0}\right) = \frac{\partial(x_1,x_2)}{\partial(x_{10},x_{20})} = \begin{pmatrix}\frac{\partial x_1}{\partial x_{10}} & \frac{\partial x_1}{\partial x_{20}}\\ \frac{\partial x_2}{\partial x_{10}} & \frac{\partial x_2}{\partial x_{20}}\end{pmatrix} = \begin{pmatrix}-\sqrt{2} & -1\\ 1 & 0\end{pmatrix}$; – local criterion for the canonicity of transformations of a Hamiltonian system into a Hamiltonian. Symplectic structure Ω defines a univalent canonical transformation (symplectomorphism).

In a general case, $\Omega^T \cdot I \cdot \Omega = cI$, where the c-valence of the canonical transformation $\Omega^T \cdot I \cdot \Omega^T \cdot I \cdot \Omega = \begin{pmatrix}0 & +\theta\\ -\theta & 0\end{pmatrix} = \begin{pmatrix}0 & -c\\ c & 0\end{pmatrix}$ => $c = -\theta$, but $|\theta| \leq 3, which\ means\ |c| \leq 3 (14)$

the valence of the canonical transformation depends on the choice of initial conditions of the dynamical system (4).

2-form ω on a Hamiltonian vector field for an arbitrary field · sets 1 form (Kozlov, 2019) $d\Psi(\cdot)$, where $\Psi(\cdot) \equiv H(\cdot)$ – Hamilton function:

$$\omega[\dot{x},\cdot] = \omega[Ax,\cdot] = (\Omega \cdot A \cdot x; \cdot) = (\Psi(k,\theta) \cdot A^{-1} \cdot A \cdot x; \cdot) = d\Psi(\cdot) \equiv dH$$

Thus, the orderly three $(\Psi; R^2; \Omega)$ will be called a Hamiltonian system on a symplectic manifold $(R^2; \Omega)$ with a symplectic structure $\omega(\Omega)$, then it is possible to define a Lagrangian system $(L; \Psi; \Omega$ with the Lagrangian of the form: $L = \frac{1}{2}\left((\Omega \cdot x\,;\dot{x}) - (\Psi(k,\theta)\cdot x;x)\right)$, where (;) is the scalar product.

Let (S; $\Psi(k,\theta)$; Ω) – be the function on a finite-dimensional symplectic manifold: $S = S = \int Ldt$, then for the isochronous variation δS the stationarity condition $\delta S \equiv 0$ specifies the extremal of the Hamiltonian action of a variational problem with "fixed ends":

$$\delta \int \left((\Omega \cdot x\,;\dot{x}) - (\Psi(\tfrac{\sqrt{2}}{2};1)\cdot x;x)\right)dt \equiv 0 \quad (15)$$

Indeed, with $\Omega = \begin{pmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}$ and $\Psi(k,\theta)=\Psi(\frac{\sqrt{2}}{2};1) = \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$

$$L = -\sqrt{2}x_1\dot{x}_1 - x_2\dot{x}_1 + x_1\dot{x}_2 - x_1^2 + 2\sqrt{2}x_1x_2 - x_2^2.$$

The condition $\delta \int Ldt \equiv 0$ is equivalent to the system of equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \end{cases} \Leftrightarrow \begin{cases} \dot{x}_1 = \sqrt{2}x_1 - x_2 \\ \dot{x}_2 = x_1 - \sqrt{2}x_2 \end{cases} \Leftrightarrow \|\dot{x}\| = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & -\sqrt{2} \end{pmatrix} \cdot \|x\| \ (16),$$

where $\begin{pmatrix} \sqrt{2} & -1 \\ 1 & -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$,

that is $\dot{x} = I \cdot \Psi(\frac{\sqrt{2}}{2};1)\cdot x = I\cdot\frac{\partial H}{\partial x}$; $H \equiv \Psi(x) -$ the Hamilton function.

$\left\|\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix}\right\| = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \left\|\begin{matrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{matrix}\right\| = \left\|\begin{matrix} -\frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_1} \end{matrix}\right\|$, which means that $x_1 \equiv p$; $x_2 \equiv q$ can be taken as canonical variables, and the variational principle built on operators (matrices) $\Omega \wedge \Psi(k,\theta)$ sets the equations of motion with a "new" operator for the original system (3) in the canonical basis (Darboux basis).

Of particular interest is the case of Morsky degeneration or Poincaré bifurcation, that is, when $\det(\Psi(k,\theta)) \equiv 0$;

$\Psi(k;\theta) = \Psi^*(k;\theta)$; $\Psi(k,\theta) = \begin{pmatrix} 2\sqrt{2}k - 1 & -\sqrt{2}\theta \\ -2k & \theta \end{pmatrix} \Rightarrow \theta = \sqrt{2}k$,

then: $\Psi(\theta) = \begin{pmatrix} 2\theta - 1 & -\sqrt{2}\theta \\ -\sqrt{2}\theta & \theta \end{pmatrix}$

The condition of degeneration takes the form:

$det(\Psi(\theta)) \equiv 0$, from which we get $\theta \equiv 0$, and, accordingly $k = \frac{3}{4}$

With these values of k and θ, operator $A(k;\theta)$ has the matrix $A = \begin{pmatrix} -2k & \theta \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 0 \\ 1 & 0 \end{pmatrix}$ −is a degenerate case, since $det(A(k;\ \theta)) \equiv 0$ as mentioned above, is an element of the semigroup with the right identity element.

Note that $\Psi_1(k,\theta)|_{\theta=0} = 0$ and so on.

However, in general, it is possible to have a case in which $A(k;\theta)$ is degenerate and $\Psi(k,\theta)$ is not – then a Poisson structure can be introduced (Treshchev, Shkalikov, 2017).

3. RESULTS

Let us introduce the notation $\tilde{A} = I \cdot \Psi(k;\theta) = I \cdot \Psi(\frac{\sqrt{2}}{2}; 1)$, where I is a symplectic unit in the canonical basis, and $\tilde{A}$ satisfies the Lie algebra (11)

Let us distinguish some spectral properties of the operator $\tilde{A}$:

1) The roots λ of the characteristic polynomial $\det(\tilde{A} - \lambda E) \equiv 0$ are symmetric relative to the real and imaginary axes on the complex plane, and, following (12), $\tilde{A} \cdot x = \pm x$, where x it the eigenvector corresponding to the eigenvalue $\lambda = \pm 1$, but the Lagrange multiplier is no longer equal to it.

2) Property 1) for operator $\tilde{A}$ entails the property of involutionality, i.e. operator $A\sim$ coincides with its inverse: $\tilde{A} \cdot \tilde{A} = \tilde{A} \cdot \tilde{A} = 1$ in matrix representation

3) $\tilde{A}^2 = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & -\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & -1 \\ 1 & -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$

A – involution in the algebra of operators in $\mathrm{M}^2(R) \equiv R^2$. (reflection characterization).

Let us consider operator $Y = \frac{\tilde{A}+1}{2}$: $Y^2 = Y$ in the construction of Y, then Y is the operator of construction and $\mathrm{M}^2(R) = kerY \oplus ImY$, therefore, - $A\sim$ is the reflection of space $\mathrm{M}^2(R)$ relative to $kerY$

The linear operator $Y = \begin{pmatrix} \frac{\sqrt{2}+1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{2}+1}{2} \end{pmatrix}$ is the idempotent in the algebra of operators $\mathrm{M}^2(R)$. Y defines the projection of the space $\mathrm{M}^2(R)$on ImY in parallel to $kerY$ ($Y^* \neq Y$ which means that Y is not an orthoprojector).

The property of involutionality of the operator $\tilde{A}$ does not entail the isometricity property: $\forall\xi;\ \eta \in \mathrm{M}^2(R)$: $(\tilde{A}\eta; \tilde{A}\xi) \neq (\eta; \xi)$, as $\tilde{A}^* \neq \tilde{A}$. Accordingly, for operator $A(k;\theta)$, the condition of self-adjacency with $k = \frac{\sqrt{2}}{2}$; θ=1: A^*=A does not entail the isometricity of A, due to its involutionality.

4) The trace of operator $\tilde{A}$ equals zero: $tr\tilde{A} \equiv 0$. Operator $div(\tilde{A}x) \equiv 0$ – the phase flux of the system (16) preserves the phase volume (measure). Let $sl_2(R)$ be a manifold of traceless operators (matrices): $\tilde{A} \in sl_2(R)$ – is a semi-simple Lie algebra, then the algebra $gl_2(R)$ has a zero center: $gl_2(R) = 0 \oplus sl_2(R)$.

Similarly, for $A(\frac{\sqrt{2}}{2}; 1)$ algebra $gl_2(R)$ degenerates into a direct sum of the center of the algebra $\xi\big(gl_2(R)\big)$ and algebra $sl_2(R)$:

$gl_2(R) = \xi\big(gl_2(R)\big) \oplus sl_2(R)$ or, respectively, for the elements of these algebras:

$$\begin{pmatrix} -\sqrt{2} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} + \begin{pmatrix} -\frac{\sqrt{2}}{2} & 1 \\ 1 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Meanwhile, elements from $\xi\big(gl_2(R)\big)$ commutate with the elements of $sl_2(R)$.

5) From property 2) follows the property of invariance of the bilinear form given by the operator (matrix) $I_{1,1}$ of the type: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and it is true that: $\tilde{A}^* \cdot I_{1,1} \cdot \tilde{A} = I_{1,1}(17)$;

and then $\tilde{A}$ belongs to the pseudo-orthogonal group $O_{p,q} = O_{1,1}$

Condition (17) in the general case for a pseudo-orthogonal group can be represented as:

$O_{ij} \cdot I_{ik} \cdot O_{kl} = I_{jl}\ or\ O^T{}_{p,q} \cdot I_{p,q} = I_{p,q}$, from which we get

$det\big(O_{p,q}\big)^2 = 1$; considering also that: $O_{i1} \cdot I_{ik} \cdot O_{k1} = I_{11} = 1 \Leftrightarrow |O_{11}| \geq 1$

Let us introduce the following notations:

– $\{O^{\uparrow}(1,1)\}$ – is the manifold of matrices for which $det\left(O^{\uparrow}(1,1)\right)^2 = 1$ and $O_{11} \geq 1$ - is a subgroup of group $O(1,1)$ and is called an orthochronous group.

– $\{O^{\downarrow}(1,1)\}$ – is the manifold of matrices for which $det\left(O^{\downarrow}(1,1)\right)^2 = 1$ and $O_{11} \leq -1$ – is a subgroup of group $O(1,1)$ and is called “parity preserving” or “orthochorous”.

– $\{SO^{\uparrow}(1{,}1)\}$ – is the manifold of matrices for which $det\left(SO^{\uparrow}(1{,}1)\right)=1$ and $O_{11}\geq 1$ – the subgroup of group $O(1{,}1)$.

Group $\{SO^{\uparrow}(1{,}1)\}$ can be defined as the intersection of $\{SO(1{,}1)\} and \{O^{\uparrow}(1{,}1)\}$or as the intersection of $\{SO(1{,}1)\} and \{O^{\downarrow}(1{,}1)\}$, as well as the intersection of the orthochronous and orthochoronous groups.

Let it be that $P=I_{1,1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$;
$T=-I_{1,1}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$; then
$PT=T\cdot P=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, while:
$\tilde{A}^{*}\cdot P\cdot\tilde{A}=P;\ \tilde{A}^{*}\cdot T\cdot\tilde{A}=T,\text{но}\ \tilde{A}^{*}\cdot(PT)\cdot\tilde{A}\neq PT.$

In the general case, $\tilde{A}=\begin{pmatrix}\sqrt{2} & -1\\ 1 & -\sqrt{2}\end{pmatrix}=\begin{pmatrix}2k & -\theta\\ \theta & -2k\end{pmatrix}\Big|_{\left(k=\frac{\sqrt{2}}{2},\ \theta=1\right)}=\begin{pmatrix}2k & \theta\\ \theta & 2k\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}\sqrt{2} & 1\\ 1 & \sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=P\circ SO^{\uparrow}(1{,}1)$, where $SO^{\uparrow}(1{,}1)=\begin{pmatrix}\sqrt{2} & 1\\ 1 & \sqrt{2}\end{pmatrix}$ – the element of the group $\{SO^{\uparrow}(1{,}1)\}$. $SO^{\uparrow}(1{,}1)=\begin{pmatrix}2k & \theta\\ \theta & 2k\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{1-k^2}} & \frac{k}{\sqrt{1-k^2}}\\ \frac{k}{\sqrt{1-k^2}} & \frac{1}{\sqrt{1-k^2}}\end{pmatrix}\Bigg|_{k=\frac{\sqrt{2}}{2}}=$
$\begin{pmatrix}\cosh(\sigma) & \sinh(\sigma)\\ \sinh(\sigma) & \cosh(\sigma)\end{pmatrix}$ (18), $where\ \tanh(\sigma)=\frac{\sinh(\sigma)}{\cosh(\sigma)}=\ =\frac{\theta}{2k}\Big|_{k=\frac{\sqrt{2}}{2}}=k=$
$\frac{\sqrt{2}}{2}, while\ \sigma -$ pseudo $-$ length, that is $\sigma -$ parameterization of a pseudocircle through its length: $cosh^2\sigma-sinh^2\sigma=1-M_1(\sigma)$manifold; or in the$(k;\theta)$ parameterization: $(2k)^2-\theta^2=1-M_2(k;\theta)$manifold (19),

where $\sigma=ln\,(\theta+\sqrt{1+\theta^2})$. The coincidence of conditions (4) and (19) gives:
$\begin{cases}\frac{\theta^2}{9}+\frac{16k^2}{9}=1\\ (2k)^2-\theta^2=1\end{cases}\Leftrightarrow\begin{cases}k^2=\frac{1}{2}\\ \theta^2=1\end{cases}$, in particular: $k=\frac{\sqrt{2}}{2};\theta=1.$

Let it be given $\{SO^{\uparrow}(1{,}1)\}$, then $SO^{\uparrow}(1{,}1)=\begin{pmatrix}\cosh(\sigma) & \sinh(\sigma)\\ \sinh(\sigma) & \cosh(\sigma)\end{pmatrix}=\cosh(\sigma)\cdot\begin{pmatrix}1 & V\\ V & 1\end{pmatrix}=\cosh(\sigma)\cdot\begin{pmatrix}1 & k\\ k & 1\end{pmatrix}\Big|_{k=\frac{\sqrt{2}}{2}}=\cosh(\sigma)\cdot\begin{pmatrix}1 & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & 1\end{pmatrix}, where\ V=tanh(\sigma)=$
$\frac{\theta}{2k}$ velocity determining the booster matrix (18)

From the relation (11) we have: $\tilde{A}=\gamma\cdot\begin{pmatrix}\sqrt{2} & -1\\ 1 & -\sqrt{2}\end{pmatrix}=\gamma\cdot\begin{pmatrix}\sqrt{2} & 1\\ 1 & \sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=$
$=\gamma\sqrt{2}\cdot\begin{pmatrix}1 & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=>\ \gamma=\frac{\cosh(\sigma)}{\sqrt{2}}$

Or in the general case $\gamma = \gamma(k, \theta);\ \gamma\left(\frac{\sqrt{2}}{2}; 1\right) = 1$

Hence, we can write down the fragments (connectivity components) that are part of the group $\{O(1,1)\}$:

$\{O(1,1)\} = \{SO^{\uparrow}(1,1)\} \oplus \{P \circ SO^{\uparrow}(1,1)\} \oplus \{T \circ SO^{\uparrow}(1,1)\} \oplus \{PT \circ SO^{\uparrow}(1,1)\}$,

and the group $\{O(1,1)\}$ is not compact as the direct sum of non-compact subgroups, in particular, $\tilde{A} \in \{P \circ SO^{\uparrow}(1,1)\} \subset \{O^{\uparrow}(1,1)\}$

Note, for example, that the following factor groups are defined:

$$\{O(1,1)\}/\{SO^{\uparrow}(1,1)\} = Z_2 \times Z_2$$

$$\{O(1,1)\}/\{SO(1,1)\} = Z_2$$

$$\{O(1,1)\}/\{O^{\uparrow}(1,1)\} = Z_2$$

And so on.

$\{SO^{\uparrow}(1,1)\}; \{SO(1,1)\}; \{O^{\uparrow}(1,1)\}$;form the normal subgroups of the group

$\{O(1,1)\}$the union of four pairwise non-intersecting adjacent classes.

The transition from operator (matrix) $\tilde{A}\ to\ SO^{\uparrow}(1,1)$ is due to the following circumstances:

First, $SO^{\uparrow}(1,1) = \Phi(k, \theta) = \Phi^{*}(k, \theta)|_{k=\frac{\sqrt{2}}{2}\ \theta=1} = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ and

$\Phi\left(\frac{\sqrt{2}}{2}; 1\right) \cdot \Psi\left(\frac{\sqrt{2}}{2}; 1\right) - \Psi\left(\frac{\sqrt{2}}{2}; 1\right) \cdot \Phi\left(\frac{\sqrt{2}}{2}; 1\right) = 0$, that is, as discussed above, the commutator $[\Phi \cdot \Psi]|_{k=\frac{\sqrt{2}}{2}\ \theta=1} \equiv 0$, therefore: $\Phi x = \mu x$ $-$integral of motion (invariant) of the system (3), whereμ $-$c is the eigenvalue for the eigenvector x of the operator (matrix) $\Phi\left(\frac{\sqrt{2}}{2}; 1\right)$.

Second, the condition $\Phi = \Phi^{*}$ opens the possibility to move from a noncompact manifold given by the operator $\tilde{A}\left(\frac{\sqrt{2}}{2}; 1\right)\ to\ \Phi\left(\frac{\sqrt{2}}{2}; 1\right)$ set on the compact (sphere $S^1 -$ i.e. to consideration of the problem on a conditional extremum for a quadratic form of the form:

$x^T \cdot \Phi(k, \theta) \cdot x \rightarrow extr\ on\ the\ sphere\ \ x^T \cdot x = 1$: (20)

$f = x^T \cdot \Phi(k, \theta) \cdot x + \mu \cdot (1 - x^T \cdot x)$, where, same as above, $f -$ Lagrange function; μ $-$Lagrange multiplier.

The stationarity condition for $\Phi\left(\frac{\sqrt{2}}{2}; 1\right)$: $\Phi\left(\frac{\sqrt{2}}{2}; 1\right) \cdot x - \mu \cdot x = 0$.

With $\mu = \sqrt{2} - 1$ we have an unconditional extremum on the sphere (minimum);

With $\mu = \sqrt{2} + 1$, there is no unconditional extremum, but with $\mu = \sqrt{2} \pm 1$ there is an extremum for $x^T \cdot \Phi\left(\frac{\sqrt{2}}{2}; 1\right) \cdot x \to min$,as the Hessian $det\left(\frac{\delta^2(x^T \Phi x)}{\delta x_i \delta x_j}\right) > 0$.

Third, the condition:

$[\Phi; \Psi] = [\Psi; \Phi]|_{k=\frac{\sqrt{2}}{2}\,\theta=1} = 0$ is satisfied by operator $\Phi = \Phi^* = \Phi(\varphi_{ij})$ of a more general kind, provided that: $\varphi_{12} = \varphi_{21}$(self-adjacency) and

$\varphi_{11} = \varphi_{22}$(the condition for the commutator to turn to zero):

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{12} & \varphi_{11} \end{pmatrix}, where\ \varphi_{ij} \in R.$$

Fourth, note that $\Phi(k, \theta) = \Phi\left(\frac{\sqrt{2}}{2}; 1\right) = SO^{\uparrow}(1,1)$ maintains the symplectic structure $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ introduced earlier:

$$\Phi^*\left(\frac{\sqrt{2}}{2}; 1\right) \cdot I \cdot \Phi\left(\frac{\sqrt{2}}{2}; 1\right) = I,$$

However, the same property is demonstrated by $\Phi = \Phi(\varphi_{ij})$: $\Phi^*(\varphi_{ij}) \cdot I \cdot \Phi(\varphi_{ij}) = I$ given that $\varphi_{11}^2 - \varphi_{12}^2 = 1(21) \Leftrightarrow det\left(\Phi(\varphi_{ij})\right) = 1$, for example,

$\Phi = \begin{pmatrix} \sqrt{3} & \sqrt{2} \\ \sqrt{2} & \sqrt{3} \end{pmatrix}$ if $det\left(\Phi(\varphi_{ij})\right) = -1, then\ \Phi^*(\varphi_{ij}) \cdot I \cdot \Phi(\varphi_{ij}) = I^{-1} \equiv I^T$, given that $\varphi_{11}^2 - \varphi_{12}^2 = -1(22)$.

It is possible to expand the class of considered operators $\Phi(\varphi_{ij})$ by excluding the condition of the commutator turning to zero, requiring only the condition of preserving the symplectic structure, $det\left(\Phi(\varphi_{ij})\right) = 1$, or the fulfillment of the condition $det\left(\Phi(\varphi_{ij})\right) = -1$,that is, for an arbitrary operator (matrix) there exists a representation π:

$\Phi(\varphi_{ij}) \to det\left(\Phi(\varphi_{ij})\right)$, which means that there is a homomorphism omX : $GL_2(R) \to R\backslash\{0\}, Mat_{2x2}(R) \to det\left(Mat_{2x2}(R)\right) :\ \forall \alpha \neq 0\ \exists \Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}, =>$ $Im(HomX) = R\backslash\{0\}$ (group) $=> GL_2(R)/SL_2(R) \cong R\backslash\{0\} \supset (Z\backslash\{0\};\times) = \{\pm 1\}\ and\ Ker(HomX) = SL_2(R)$.

Introducing the notation for the group that preserves the symplectic structure $\{G_1\} -$ a set of matrices (operators) with $detG_1 = 1$ and, consequently - $\{G_2\}$ with $detG_2 = -1$, we receive the following:

The group $\{G_1\}$ includes, for example, the elements: $\Phi(\varphi_{ij}); PT$, and $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix} = P$ with $\alpha == -1; h = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, etc.

To the group $\{G_2\}$we can attribute: $A\left(\frac{\sqrt{2}}{2};1\right); \tilde{A}\left(\frac{\sqrt{2}}{2};1\right); P$, while $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix} = P\ at\ \alpha = 1;\ T; l = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, etc.$

Fifth, consider a restriction to Z of relations (21) and (22): $\varphi_{11}^2 - \varphi_{12}^2 = \pm 1$, given on R, for the special case: $a^2 - 2b^2 = \pm 1\quad a, b \in Z$(Pell's equation) (van der Waerden, 1976).

In this case the recurrence relations are fulfilled:

$$\begin{cases} a_{n+1} = a_n + 2b_n \\ b_{n+1} = a_n + b_n \end{cases}$$

This sets up a linear automorphism $AutGL_2(Z) : R^2 \to R^2$ of the two-dimensional real plane (Gorbatsevich, 2004) ($GL_2(Z)$ is a subgroup in $GL_2(R)$. The linear transformation $GL_2(Z)$ retains the integer lattice $Z^2 \subset \mathrm{R}^2$ and induces the automorphism of the two–dimensional torus $T^2 \equiv \mathrm{R}^2/Z^2, - DiffA :\ T^2 \to T^2$;

$DiffA|_{GL_2(Z)} = d_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ is Thom's example (Gorbatsevich, 2004).

d_1 is presentation of$DiffA$(Anosov's linear diffeomorphism). Since $det(d_1) = -1, then\ d_1 \in \{G_2\}$:

$d_1^* \cdot I \cdot d = I^{-1}; \det(d_1 - \lambda E) = 0 <=> \lambda(d_1) = 1 \pm \sqrt{2}$, although for $\mu\left(\Phi\left(\frac{\sqrt{2}}{2};1\right)\right) = 1 + \sqrt{2}$ unconditional extremum (extremum on the compact) was absent.

Similarly, by the action on the basis elements of the lattice $\frac{R^2}{Z^2}$ in her fundamental field, one can take $DiffA|_{GL_2(Z)} = d_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - Arnold's\ cat$ (Arnold, 2012; Gorbauevich, 2022).

$d_2 \in \{G_1\} : \det(d_2) = 1, и\ d_1^* \cdot I \cdot d = I.$

The composition of groups $\{G_1\}\ and\ \{G_2\}$ allows speaking of topologically adjoint diffeomorphisms, i.e. the existence of geomorphisms; for instance, $h\ and\ PT :\ d_2(h) = h^{-1} \cdot d_2 \cdot h = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}, d_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, i.e. matrices d_2 and $d_2(h)$ have the same characteristic polynomial, and therefore define the same operator $DiffA$, with $d_2(PT) = (PT)^{-1} \cdot d_2 \cdot (PT) \equiv d_2$

The above circumstances allow, on the one hand, to turn to the issues of structural stability (roughness) of linear dynamic systems given by Anosov diffeomorphisms on compact manifolds forming an open subset in the group of all diffeomorphisms on a given manifold. On the other hand, this allows for the use of the methods of hyperbolic dynamics, - considering the matrix of the linearization operator as hyperbolic, and the fixed point as hyperbolic. Note that if, for example d_1 and d_2 belong to the group $GL_2(Z)$, they are hyperbolic when and only when d_1^2 (d_2^2), belonging to the group $SL_2(Z)$,are hyperbolic (Gorbatsevich, 2004).

For the hyperbolic point X_0,i.e. for any element $G_1 from \{G_1\}$ and G_2 $from$ $\{G_2\}, -det\ (G_i \pm E) \neq 0$ there are a stable $W^S(x_0)$ and an unstable $W^u(x_0)$ manifolds, meanwhile $W^u(x_0)$ is stable with respect to the inverse diffeomorphism. $W^S(x_0)$ and $W^u(x_0)$are homeomorphic to $R^{dimW^S(x_0)}$ and $R^{dimW^u(x_0)}$(in the internal topology).

Six, it becomes possible to analyze the linearization operators by topological dimensionality (the type of the equilibrium position point).

Operator $A\left(\frac{\sqrt{2}}{2};1\right)$ c $det\left(A\left(\frac{\sqrt{2}}{2};1\right)\right) = -1 < 0$ defines the saddle fixed point; this is also the behavior of the operator $\tilde{A}\left(\frac{\sqrt{2}}{2};1\right) \equiv P \circ SO^{\uparrow}(1,1) -$ saddle point, and $T \circ SO^{\uparrow}(1,1)$ $-$works similarly.

Ifx_0 is the saddle point, its stable and unstable manifolds have nonzero topological dimensionality:

$dimW^u(x_0) = 1, -W^u(x_0)\backslash\{x_0\}$ – one-dimensional unstable separatrix;

$dimW^s(x_0) = 1, -W^s(x_0)\backslash\{x_0\}$ $-$ one-dimensional stable separatrix. The saddle point is also characteristic of the Anosov diffeomorphism d_1.

Operator $A(k;\theta) = A\left(\frac{\sqrt{2}}{2};-1\right) = \Omega$ – defines a stable focus at the point of equilibrium position, and since Ω is not nondegenerate, then there is $A'\left(\frac{\sqrt{2}}{2};1\right) = \begin{pmatrix} -1 & -\sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}$ that ensures that $A\left(\frac{\sqrt{2}}{2};1\right) \cdot A'\left(\frac{\sqrt{2}}{2};1\right) = \Omega^{-1}$. $A'\left(\frac{\sqrt{2}}{2};1\right)$ determining the saddle point, represents the operator $Diff(R^2)$, set on torus $T^2 = R^2/Z^2$ by matrix

$d_1^{-1}\left(\frac{\sqrt{2}}{2};1\right)$, while $d_1 \cdot d_1^{-1} = E$, and homeomorphism $R^2 \to T^2 = R^2/Z^2$can be constructed.

For operator $SO^{\uparrow}(1,1) = \Phi\left(\frac{\sqrt{2}}{2};1\right)$. $tr\left(\Phi\left(\frac{\sqrt{2}}{2};1\right)\right) > 0, but\ 0 < det\left(\Phi\left(\frac{\sqrt{2}}{2};1\right)\right) < \left(\frac{tr\left(\Phi\left(\frac{\sqrt{2}}{2};1\right)\right)}{2}\right)^2$ – the unstable node $-dimW^u(x_0) = 2$.

The diffeomorphism d_2 has $det(d_2) = 1, while 0 < det(d_2) < \left(\frac{tr(d_2)}{2}\right)^2$ and $tr(d_2) > 0$, i.e. by formal parameters, it sets an unstable node. However, being a diffeomorphismon torus $T^2 = \frac{R^2}{Z^2}$, it defines its dense irrational winding, defining an unstable manifold $W^u(x_0)$ with dimensionality $dimW^u(x_0) = 1$ in the invariant direction of the eigenvector $\left\|\begin{matrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{matrix}\right\|$, adjoint to the eigenvalue $\lambda = \frac{3+\sqrt{5}}{2}\ for\ d_2$; and a stable manifold $W^s(x_0)$ with dimensionality $dimW^s(x_0) = 1$ in the orthogonal direction with $\lambda = \frac{3-\sqrt{5}}{2}\ for\ d_2$.

Operator (matrix) $PT \circ SO^{\uparrow}(1,1) -$ defines the stable node at the equilibrium position: $det\left(PT \circ SO^{\uparrow}(1,1)\right) = 1, tr\left(PT \circ SO^{\uparrow}(1,1)\right) < 0$

$$0 < det\left(PT \circ SO^{\uparrow}(1,1)\right) < \left(\frac{tr\left(PT \circ SO^{\uparrow}(1,1)\right)}{2}\right)^2.$$

In the general case the bifurcation diagram for the operator $A(k;\theta) = \begin{pmatrix} -2k & \theta \\ 1 & 0 \end{pmatrix}$ can be represented as presented below (Fig. 3):

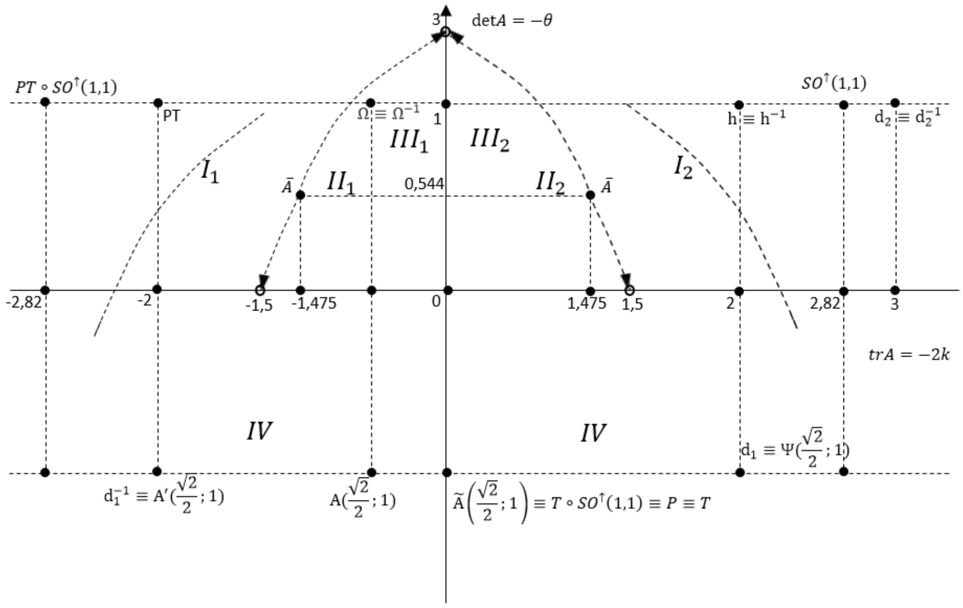

Figure 3. Bifurcation diagram ($I_1 -$ stable node ; $I_2 -$ unstable node ; $II_1 -$ dicritical (degenerate) stable node ; $II_2 -$ dicritical (degenerate) unstable node ; $III_1 -$ stable focus ; $III_2 -$ unstable focus ; $IV -$ saddle)

Since $det\big(A(k;\theta)\big) = -\theta$ and $tr\big(A(k;\theta)\big) = -2k$ and $k > 0, then\ \bar{A},$ a degenerate stable node (unstable node), satisfies the condition

$$\begin{cases} -\theta = k^2 \\ \frac{\theta^2}{9} + \frac{16k^2}{9} = 1 \end{cases} \Leftrightarrow \begin{cases} \bar{\theta} = -\bar{k}^2 = -\big(\sqrt{73} - 8\big) \approx -0{,}544 \\ 2\big|\bar{k}\big| = 2\sqrt{\sqrt{73} - 8} \approx 1{,}475 \end{cases}$$

Similar conditions of dicrity (degeneracy) are satisfied by PT (stable node) and h (unstable node).

As noted previously and apparent from the diagram, d_1^{-1} and $A'(\frac{\sqrt{2}}{2}; 1)$ are representatives of the same operator, but the same property of the invariance of the characteristic polynomial is demonstrated by d_1 and $\Psi(\frac{\sqrt{2}}{2}; 1)$, which means that d_1 can also be assigned the meaning of the quadratic invariant on the torus.

4. CONCLUSION

The two operators (matrices) $A(k;\theta)$ and $\Psi(k;\theta)$, which satisfy the condition of nondegeneracy (openness condition on manifold), as well as the presence of a quadratic invariant $\Psi(x)$ allows us to formulate the problem of toric topology: toric compactification - invariance of torus action on a phase space (noncompact manifold) and Hamiltonian toric manifold $-$ compact connected Hamiltonian T^2 $-$manifold (R^2; ω; *M*) with the effective action of the torus T^2, where $M -$ the mapping of moments, i.e. the mapping to the Lie algebra of the torus T^2 (Buchstaber, Panov, 2015; 2020).

Let us consider the aspects related to the question of gyroscopic stabilization in a mechanical system.

For this purpose let us introduce an "operator" representation of the dynamic system (1):

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\ddot{x} + \begin{pmatrix}2k & 0\\0 & 2k\end{pmatrix}\dot{x} + \begin{pmatrix}-\theta & 0\\0 & -\theta\end{pmatrix}x \equiv 0 \ (23)$$

Indeed, system (1) can be presented in the form:

$$\begin{cases}\ddot{\Omega}_{orb} + 2k\dot{\Omega}_{orb} = 3cos2\varepsilon\Omega_{orb}\\ \ddot{\varepsilon} + 2k\dot{\varepsilon} = \frac{3}{2}sin2\varepsilon - 2k\end{cases} (24)$$

By laying out the right parts in a row by degrees $(\Omega_{orb} - \Omega_0)\ and\ (\varepsilon - \varepsilon_0)$ in the vicinity of the equilibrium position and minding the initial conditions (4) we get (23), where $M = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ – the inertia operator,

$-M^* \equiv M;\quad D^* \equiv D = \begin{pmatrix}2k & 0\\0 & 2k\end{pmatrix}$ – dissipative force operator, while $D \geq 0$

$\Pi^*(\theta) \equiv \Pi(\theta) = \begin{pmatrix}-\theta & 0\\0 & -\theta\end{pmatrix}$ – the conservative force operator.

Equation (23) on the solutions (3) is presented in the form (Kozlov, 2021; 2022; de Leon, Rodrigues, 1989):

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\ddot{x} + \begin{pmatrix}-4k^2 - \theta & 2k\theta\\2k & -\theta\end{pmatrix}x = 0 \ (25),$$

Herein the condition $\Pi(k;\theta) \equiv \Pi(k;\theta)$ corresponds to the condition $\theta \equiv 1$(initial conditions).

Let it be that $\Pi(k;\theta) -$the conservative force operator defining a quadratic form (potential energy) of the following form:

Let it be that $\Pi(k;\theta) -$ the conservative force operator defining a quadratic form (potential energy) of the following form:

$$\frac{1}{2}(\Pi x; x) \equiv \frac{1}{2}(\Pi(k;\theta)\cdot x; x) = \frac{1}{2}\left(\Pi\left(\frac{\sqrt{2}}{2}; 1\right)\cdot x; x\right) =$$

$= \frac{1}{2}\left(\begin{pmatrix}-3 & \sqrt{2}\\\sqrt{2} & -1\end{pmatrix}\cdot \left\|\begin{matrix}x_1\\x_2\end{matrix}\right\|; \left\|\begin{matrix}x_1\\x_2\end{matrix}\right\|\right) = \frac{1}{2}\left(-3x_1^2 + 2\sqrt{2}x_1x_2 - x_2^2\right)$, which can be transformed by a nondegenerate transformation to the form:

$\frac{1}{2}\left(-\left(2-\frac{2\sqrt{2}}{\sqrt{3}}\right)z_1^2-\left(2+\frac{2\sqrt{2}}{\sqrt{3}}\right)z_2^2\right)$, where $p \equiv i^-$ is the degree of oddness of the quadratic form $\frac{1}{2}(\Pi \cdot x; x), i^- = 2 -$ the negative inertia index of the form, since p is even and $D = 0$(dissipative force operator), then the system can have gyroscopic stabilization with the operator

$\Gamma : \Gamma^* = -\Gamma$, for example, considering equation (23) based on the solutions

$\dot{x} = \tilde{A} \cdot x \equiv \left(P \circ SO^{\uparrow}(1{,}1)\right) \cdot x$ or $\dot{x} = \left(T \circ SO^{\uparrow}(1{,}1)\right) \cdot x$, we get that for

$\Gamma = I \; and \; \Gamma = -I$ respectively, provided that $\Omega^T \cdot \Gamma \cdot \Omega = \Gamma$ is the valence of the canonical transformation $c = \pm 1 \; (see\ (14))$.

This article indicates only the fundamental possibility of gyroscopic stabilization of a linear dynamic system, and a detailed analysis was not carried out, but it seems very promising and prospective from the energy point of view.

Finally, let us note the following two facts:

1. The operator $A(k;\theta)$ satisfies the following matrix equation:$(A + D) \cdot A + \Pi \equiv 0 \; \forall\theta$ while $det(D^2 - 4\Pi) > 0 \; \forall k; \theta$, so there is a "compressive" operator

$A = -D^{-1} \cdot (A^2 + \Pi) \; with\ the\ solution A' = -D + \Psi(k;\theta)$ satisfying the equation $(A' + D) \cdot A' + \Pi = 0$

$A' = A'(\Psi(k;\theta))|_{\substack{k=\frac{\sqrt{2}}{2} \\ \theta=1}} = \begin{pmatrix} -1 & -\sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}$ noted above.

The existence of $A'\left(\frac{\sqrt{2}}{2};1\right)$ entails the fulfillment of two fundamental relations:

$$A'\left(\frac{\sqrt{2}}{2};1\right) \cdot \Psi\left(\frac{\sqrt{2}}{2};1\right) = E \; and \; \left(A\left(\frac{\sqrt{2}}{2};1\right) \cdot A'\left(\frac{\sqrt{2}}{2};1\right)\right)^{-1} = \Omega$$

Geometrically, the "operator" equation (23) is equivalent to the two equations: $\dot{x} = A \cdot x \; and \;\; \dot{x} == A' \cdot x$, defining two invariant planes $\Delta_1 \subset R^2$ and $\Delta_2 \subset R^2$, which are orthogonal to each other and are the direct sum of the original phase space: $R^4 = \Delta_1 \oplus \Delta_2$

2. Presenting (25) in the form $\ddot{x} = -\Pi(k,\theta) \cdot x$, and noting that

$A^2(k,\theta) = -\Pi(k,\theta)$, given that $\theta \neq 0$;

$\Psi(x;\dot{x}) = \frac{1}{2}(A^{-1}(k,\theta) \cdot \dot{x};\dot{x}) - \frac{1}{2}(A(k,\theta) \cdot x; x)$ the quadratic invariant:

$\Psi(x;\dot{x})|_{\dot{x}=A(k,\theta)\cdot x} \equiv 0$ in accordance with (12).

Similarly, $-\Pi(k,\theta) = \begin{pmatrix} 4k^2+\theta & -2k\theta \\ -2k & \theta \end{pmatrix}$ with $\theta \neq 1$ can be represented as: $-\Pi(k,\theta) = \Pi_1(k,\theta)\cdot\Pi_2(k,\theta)$, while $det(\Pi_1(k,\theta)) \neq 0$; $\Pi_1{}^*(k,\theta) = \Pi_1(k,\theta)$; $\Pi_2{}^*(k,\theta) = \Pi_2(k,\theta)$;

then $-\Pi(k,\theta) = A(k,\theta)\cdot A(k,\theta) = \Pi_1(k,\theta)\cdot\Pi_2(k,\theta)$

$\Psi(x;\dot{x}) = \frac{1}{2}(\Pi_1^{-1}(k,\theta)\cdot\dot{x};\dot{x}) - \frac{1}{2}(\Pi_2(k,\theta)\cdot x;x)$ - the quadratic invariant:

$$\Psi(x;\dot{x})|_{\dot{x}=A(k,\theta)\cdot x} \equiv 0$$

For example, let it be that $\Pi_2(k,\theta) = \Phi\left(\frac{\sqrt{2}}{2};1\right)$ then

$-\Pi(k,\theta) = \Pi_1(k,\theta)\cdot\Pi_2 = \Pi_1(k;-\sqrt{2}k+1)\cdot\Pi_2$, according to equation (4): $\begin{cases}\theta^2+16k^2=9\\ \theta=-\sqrt{2}k+1\end{cases}$ and condition (20) for $\Phi\left(\frac{\sqrt{2}}{2};1\right)$.

We can also introduce a quadratic invariant of the form (Kozlov, 2020; Galiullin, 1988):

$\Psi(x;\dot{x}) = \frac{1}{2}((\Pi_1 A)^{-1}\cdot\dot{x};\dot{x}) - \frac{1}{2}\cdot((\Pi_2 A^{-1})^{-1}\cdot x;x)$, so that $\Psi(x;\dot{x})|_{\dot{x}=\Pi_1(k,\theta)\cdot x} \equiv 0$, under the same condition $A^2 = \Pi_1\cdot\Pi_2 = -\Pi$, whose natural generalization leads to the possibility of constructing a Lie algebra (Mimura, Toda, 1991; Perelomov, 1990) $\Pi_1\cdot\Pi_2 + \Pi_2{}^T\cdot\Pi_1 = 0$, provided solvability of the system: $\begin{cases}-\Pi = \Pi_1\cdot\Pi_2\\ -\Pi = -\Pi_2{}^T\cdot\Pi_1\end{cases}$

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