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ACCURATE RADIAL BASIS FUNCTIONS TECHNIQUE FOR COMPETITIVE AND EFFICIENT SOLUTIONS OF NON-LINEAR BLACK-SCHOLES EQUATIONS

TÉCNICA DE FUNÇÕES DE BASE RADIAL ACURADA PARA SOLUÇÕES COMPETITIVAS E EFICIENTES DE EQUAÇÕES NÃO LINEARES DE BLACK-SCHOLES

TÉCNICA DE FUNCIÓN DE BASE RADIAL ACURADA PARA SOLUCIONES COMPETITIVAS Y EFICIENTES DE ECUACIONES NO LINEALES DE BLACK-SCHOLES

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Abstract

Objective: This article aims to solve the non-linear Black Scholes (BS) equation for European call options using Radial Basis Function (RBF) Multi-Quadratic (MQ) Method.

Methodology / Approach: This work uses the MQ RBF method applied to the solution of two complex models of nonlinear BS equation for prices of European call options with modified volatility. Linear BS models are also solved to visualize the effects of modified volatility. Additionally, an adaptive scheme is implemented in time based on the Runge-Kutta-Fehlberg (RKF) method.

Originality / Relevance: The original computational scheme of a solver containing an efficient integrator of the diffusional term is presented. The adaptive, accurate and coherent temporal integrator, chosen, was the RKF method.

Main Results: The numerical results obtained, when compared with the solutions of Sevcovic and Zitnanská (2016) and Ankudinova (2008), allow us to state that the RBF method is accurate and easy to program. Adaptive methods over time proved to be efficient both in terms of speed (number of iterations to reach the final simulation time) and in terms of accuracy in relation to the results without the implementation of the method.

Theoretical / methodological contributions: This paper presents an original numerical solver for nonlinear financial problems. Part of the solver structure was built based on two analytical-numerical solvers available in the literature, a modified RBF solver and an RKF temporal integrator. The solver performs excellently when compared to other available models.

Keywords: Non-linear Black Scholes Equation. Radial Basis Functions. Adaptive Method. Option Pricing.

Resumo

Objetivo: o presente artigo tem como objetivo solucionar a equação de Black Scholes (BS) não linear para opções de compra europeias por meio do método de funções de base radial (FBR) Multiquádrica (MQ) com adaptatividade temporal.

Metodologia / Abordagem: Este trabalho utiliza o método FBR MQ aplicado à solução de dois modelos complexos de equação não linear de BS para preços de opções de compra europeias com volatilidade modificada. Modelos lineares de BS também são resolvidos para visualizar os efeitos da volatilidade modificada. Adicionalmente, implementa-se um esquema adaptativo no tempo tendo por base o método de Runge-Kutta-Fehlberg (RKF).

Originalidade / Relevância: Apresenta-se um esquema original computacional de um resolvedor que contém um integrador eficiente do termo difusivo. O integrador temporal adaptativo, acurado e coerente, escolhido, foi o método de RKF.

Principais Resultados: Os resultados numéricos obtidos, quando comparados com a literatura, permitem afirmar que o método FBR é preciso e de fácil programação. Os métodos adaptativos no tempo mostraram-se eficientes quer em termos de rapidez (número de iterações para atingir o tempo final de simulação) quanto em termos de acurácia em relação aos resultados sem a implementação do método.

Contribuições teóricas / metodológicas: Este trabalho apresenta um solucionador numérico original para problemas financeiros não lineares. Parte da estrutura do solucionador foi construída com base em dois solucionadores analítico-numéricos disponíveis na literatura, um solucionador de RBF modificado e um integrador temporal de RKF. O solucionador apresenta desempenho excelente, quando comparado com outros modelos disponíveis.

Palavras-chave: Equação de Black Scholes não linear. Funções de Base Radial. Método Adaptativo. Precificação de Opções.

Resumen

Objetivo: este artículo tiene como objetivo resolver la ecuación no lineal de Black Scholes (BS) para opciones de compra europeas utilizando el método de función de base radial (FBR) multi-cuadrática (MQ).

Metodología / Enfoque: Este trabajo utiliza el método FBR MQ aplicado a la solución de dos modelos de ecuaciones BS no lineales para precios de opciones de compra europeas con volatilidad modificada, demostrado en los trabajos de Sevcovic y Zitnanská (2016) y Ankudinova (2008). También se resuelven modelos lineales de BS para visualizar los efectos de la volatilidad modificada. Además, se implementa un esquema adaptativo en el tiempo basado en el método Runge-Kutta-Fehlberg (RKF).

Originalidad / Relevancia: Se presenta el esquema computacional original de un solucionador que contiene un integrador eficiente del término difusor. El integrador temporal adaptativo, preciso y coherente, elegido, fue el método RKF.

Resultados principales: Los resultados numéricos obtenidos, al compararlos con las soluciones de Sevcovic y Zitnanská (2016) y Ankudinova (2008), nos permiten afirmar que el método FBR es preciso y fácil de programar. Los métodos adaptativos a lo largo del tiempo demostraron ser eficientes tanto en términos de velocidad (número de iteraciones para alcanzar el tiempo de simulación final) como en términos de precisión en relación a los resultados sin la implementación del método.

Contribuciones / metodológicas: Este trabajo presenta un solucionador numérico original para problemas financieros no lineales. Parte de la estructura del solucionador se construyó sobre la base de dos solucionadores análonos-numéricos disponibles en la literatura, un solucionador RBF modificado y un integrador temporal RKF. El solucionador funciona excelentemente en comparación con otros modelos disponibles.

Palabras clave: Ecuación no lineal de Black Scholes. Funciones de base radial. Método adaptativo. Precio de las opciones.

1. INTRODUCTION

Derivative pricing is one of the most important processes in financial markets (Milovanovic, 2018). Financial derivatives are becoming increasingly popular nowadays, not only as instruments of hedging but also for speculative transactions (Janková, 2018). Basic types of derivatives include futures, options, and forwards (Wilmott, 2007).

There are several models to predict the price of an option. The formula proposed by Fischer Black and Myron Scholes in 1973, known as the Black-Scholes (BS) model, is the most widely used one (Fall, Ndiaye & Sene, 2019).

The basic equation of BS is a linear parabolic hyperbolic equation with stochastic and deterministic parameters and variables. Improvements in the original model lead us to a set of partial differential equations essentially equivalent to the diffusion and convection equation used in engineering (Wilmott, 1998). The linear equation of BS with constant volatility was derived under several restrictive assumptions such as frictionless, liquid, complete markets, etc. (Grossinho, 2017). However, the ideal conditions that make volatility constant do not occur due to the effects of transaction costs, investor preferences, incomplete markets, effects due to the large number of traders in the market, among others (During, 2005).

The literature is abundant in models used to estimate volatility (Lin, Li and Wu 2018). Typically volatility models considers the effects of the market. These models result in strong nonlinear parabolic diffusion-convection BS equations (During, 2005). Many numerical methods have been proposed to approximate the solution of convective-diffusive equations added to the nonlinear term of modified volatility, from which the nonlinear BS equation originates. These methods include finite differences methods, finite elements, finite volumes, and contour elements that originate from local interpolation schemes and require the use of meshes. Solutions of finite differences and finite elements for the convective-diffusive equation present numerical problems of oscillation and damping (Amster, Averbuj & Mariani, 2003; Boztosun & Charafi, 2002; Hoffman, 1992; Lee, Peraire & Zienkiewicz, 1987; Murphy & Prenter, 1985; Tomas III & Yalamanchili, 2001; Zienkiewicz & Taylor, 1991; Wilmott, 1998; Wilmott, Howison & Dewynne, 1995 as quoted in Santos, Souza & Fortes, 2009).

Based on the above analysis it can be inferred that the market, by the large number of models being made available, has not yet solidified into a high confidence model. In other words, models are still restrictive subject to a small range of applications. Therefore, we propose in this work a numerical model that allows expending the investigation range leading to more accurate results.

More specifically this work aims to:

1. Evaluate the behavior of numerical solutions of the nonlinear Black-Scholes equation for European call options subject to the modified volatility fun.
2. Through multiquadric radial bass functions (RBF) (MQ).

Through two nonlinear models of modified volatility:

a) Variable transaction cost model proposed in the work of Sevcovic and Zitnanská (2016).
b) Identity Model proposed by Barles and Sonner (1998) with numerical results presented in the work of Ankudinova (2008).

3. Investigate by mesh refinement the non-linearity regions of the inflection point of the payoff function.
4. Compare the results obtained from nonlinear models with linear models.
5. Implement a simple and efficient method of temporal adaptive solution of partial differential equations, applied, together with the RBF technique for solving problems of linear and nonlinear options.
6. Perform a sensitivity and error analysis for the methods and problems proposed in the above items.

2. THEORETICAL FRAMEWORK

Nonlinear Black-Scholes Equation

The widely accepted mathematical model for evaluating the temporal value of a *V(S,t)* option is the Black-Scholes equation. It is based on a stochastic model for the behavior of the asset price *(S)*, whose solution leads to the current price *V(S,0)* of an option that expires at the end time *T* (Meyer, 1998). The classic form of the basic equation of Black-Scholes or BS is (Wilmott, 1998):

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV = 0 \qquad (1)$$

where V, t, σ, S *e* r are, respectively, the value of the option, time, volatility, price of the asset (underlying asset – stochastic variable), and risk-free interest rate.

The asset value *(S)* follows a Brownian geometric movement which means that if $W = W(t)$ is a standard Brownian movement, then *(S)* satisfies the following stochastic differential equation (SDE) (Wilmott, 1998):

$$dS = \mu S dt + \sigma S dW \tag{2}$$

where μ represents the average rate of asset price growth.

However, the assumption listed above does not represent reality. In this way, several options pricing models in which the volatility function σ is no longer constant have been developed. The classical model should be represented as a nonlinear equation, in which both volatility σ as the coefficient μ should depend on the time *t*, the price of the asset *S* or the value of option *V* itself (Ankudinova, 2008).

In this work, we focus in the case in which volatility depends on the second derivative of the price of option *V(S,t)* in relation to the price of the asset *S* (Gama) that can be written as (Sevcovic & Zitnanská, 2016):

$$V_t + \frac{1}{2}\tilde{\sigma}(SV_{ss})^2 S^2 V_{ss} + rSV_s - rV = 0 \tag{3}$$

where $\tilde{\sigma}(SV_{ss})^2$ is product function of the asset price by Gamma (second derivative from the price of option *V* in relation to the price of the asset *S*).

The main motivation to solve the nonlinear equation of BS with the volatility function of $\tilde{\sigma}(SV_{ss})^2$is due to the pricing of more realistic options in which one can take into account the presence of transaction costs, market feedbacks, risks from unprotected portfolio and other effects (Duris, Tan, Lai & Sevcovic, 2015).

The Value of *V(S,t)* of a European call option is obtained by solving the Equation (3) at $0 \leq S < \infty$ and $0 \leq t \leq T$, considering the following boundary conditions:

$$\begin{aligned} &V(S,T) = (S-K)^+ \ 0 \leq S < \infty \\ &V(0,t) = 0 \quad 0 \leq t < T \\ &V(S,t) \sim S - Ke^{-r(T-t)} \quad S \to \infty \end{aligned} \tag{4}$$

where *K* is the exercise price of the option.

It should be noted that the first condition refers to the payment function, that is, the value of a call option at maturity (t =T).

Nonlinear Black-Scholes Equation - Variable Transaction Cost Model

Due to the variation in the number of transactions made by several investors in the market, the costs of these transactions may fluctuate, since the higher the number of shares that a same investor trades the less he will pay for the transactions. Sevcovic and Zitnanská (2016) presented in their work a generalization of the Leland (1985) model taking into account in the calculation of the transaction cost the volume of transactions negotiated $\delta(\Delta\delta > 0$ ou $\Delta\delta < 0)$.

Considering the term r_{CT} as a measure of the expected value of the transaction cost change per unit of time interval Δt, a generalization of the Black Scholes equation is obtained as below:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV - r_{CT}S = 0 \tag{5}$$

The term r_{CT} obtained through the Leland equation for the constant transaction cost case is defined as:

$$r_{CT} = \frac{1}{2}\sigma^2 S Le|V_{ss}| \tag{6}$$

where $Le = \sqrt{\frac{2}{\pi}}\frac{C_0}{\sigma\sqrt{\Delta t}}$ and C_0 is the constant transaction cost not dependent on the volume of transactions traded $\Delta\delta > 0$ ou $\Delta\delta < 0$.

Similarly, taking into account in the calculation of the transaction cost the volume of transactions traded ($\Delta\delta > 0$ ou $\Delta\delta < 0$), the term r_{CT} is affected by a variable transaction cost function $\tilde{C}(\alpha)$ dependent on the product of S and gamma (V_{ss}). In this way, the r_{CT} in Equation (6), can be rewritten as follows:

$$r_{CT} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\tilde{C}(\alpha)\alpha}{\Delta t} \tag{7}$$

where $\alpha = \sigma S|V_{ss}|\sqrt{\Delta t}$

By replacing Equation (7) in Equation (5) a generalization of the Leland equation is obtained for the case of variable transaction costs. Consequently, the term $\tilde{\sigma}(SV_{ss})^2$ in Equation (3) can be defined as:

$$\tilde{\sigma}(SV_{ss})^2 = \sigma^2(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma S|V_{SS}|\sqrt{\Delta t})\frac{sgn(V_{SS})}{\sigma\sqrt{\Delta t}} \tag{8}$$

In fact, for a small volume of assets traded (ξ), the rate of transaction costs follows a value C_0. According to Sevcovic e Zitnanská (2016), when the volume is large enough, a discount is applied with a transaction cost fee $\bar{C}_0 \leq C_0$ lower. From the use of the integration by parts, it can be inferred that the modification of the value is a result of a function of transaction costs by parties given by:

$$\tilde{C}(\xi) = C_0 - \kappa\xi \int_{\frac{\xi_-}{\xi}}^{\frac{\xi_+}{\xi}} e^{\frac{-x^2}{2}} dx,\ \xi \geq 0 \tag{9}$$

where ξ_+e ξ_- are the intervals for part integration.

Nonlinear Black-Scholes Equation - Barles and Soner Identity Model (1998)

Barles and Soner (1998) created a variable volatility model based on the utility function of Hodges and Neuberger (1989), which seeks to simulate the behavior of the investor in the market. Therefore, for the pricing of a European type option, the modified volatility is defined as follows:

$$\tilde{\sigma}^2 = \sigma^2(1 + \psi(e^{r(T-t)}a^2S^2V_{SS})) \tag{10}$$

$$a = \kappa/\sqrt{\varepsilon} \tag{11}$$

$$\varepsilon = \frac{1}{\gamma N} \tag{12}$$

where κ is the proportional transaction cost, γ represents the risk aversion factor and N is the number of options. The utility function $\psi(x)$ is the solution to the following nonlinear ordinary differential equation (ODE):

$$\psi'(x) = \frac{\psi(x)+1}{2\sqrt{x\psi(x)-x}} \tag{13}$$

with the following initial condition,

$$\psi(0) = 0 \tag{14}$$

An analysis of the differential equation, Equation (13), proposed by Barles and Soner (1998) implies that:

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1\ e\ \lim_{x\to-\infty} \frac{\psi(x)}{x} = -1 \tag{15}$$

The property presented in Equation (15) allows us to treat $\psi(x)$ as identity for large arguments thus allowing to simplify calculations. Consequently, the term $\tilde{\sigma}(SV_{SS})^2$ in equation (3) can be defined as:

$$\tilde{\sigma}(SV_{SS})^2 = \sigma^2(1 + e^{r(T-t)}a^2S^2V_{SS}) \tag{16}$$

3. METHODOLOGY

Radial Base Functions Method

The Radial Base Functions Method uses linear combinations of a base function $\phi(\mathrm{r})$ of a variable, expanded over a given scattered data center $\mathrm{S}_i \in \Re^d, i = 1, \ldots, N$ to approximate an unknown function $\mathrm{V(S,t)}$ by:

$$\mathrm{V(S,t)} = \sum_{j=1}^{N} \lambda_j(\mathrm{t})\phi(r_j) = \sum_{j=1}^{N} \lambda_j \phi(\|S_i - S_j\|) \tag{17}$$

where $r_j = \|S_i - S_j\|$ is the Euclidean norm and λ_j are the coefficients to be determined.

The most commonly used types of radial base functions are (Martin & Fornberg, 2017):

$$\text{Multiquadric, } \phi(r_j) = \sqrt{c^2 + r_j^2} \tag{18}$$

$$\text{Gaussian, } \phi(r_j) = e^{-cr_j^2} \tag{19}$$

$$\text{Polyharmonic Splines (PHS), } \phi(r\_j\ ) = \begin{cases} r_j^m, m\ \acute{\imath}mpar \\ r_j^m log r_j, m\ par \end{cases} \tag{20}$$

where c and m are shape parameters.

The RBF methodology to obtain the numerical solution of the BS equation requires the discretization of equation (3) together with the volatility term given by Equations (8) and (16). Thus, the nonlinear Black-Scholes equation can be discretized as:

$$\frac{\partial V(S,t)}{\partial t} = f(S,t) \approx \frac{1}{2}\tilde{\sigma}(SV_{ss})^2 S^2 V_{ss} + rSV_s - rV \tag{21}$$

Replacing the approach to $\mathrm{V(S,t)}$ given by Equation (17) added to the volatility model given by Equation (8), one can write equation (21) as follows:

$$\frac{\partial\lambda(t)}{\partial t}\Phi(\mathrm{r}) \approx \frac{1}{2}\sigma^2\left[(1 - \sqrt{\frac{2}{\pi}}\tilde{C}\left(\sigma S\left|\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t)\right|\sqrt{\Delta t}\right)\frac{sgn\left(\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t)\right)}{\sigma\sqrt{\Delta t}}\right]S^2\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t) +$$

$$+ rS\frac{\partial\Phi(\mathrm{r})}{\partial S}\lambda(t) - r\Phi(\mathrm{r})\lambda(t) \tag{22}$$

Multiplying both sides of the Equation (22) by $\Phi(\mathrm{r})^{-1}$ we have:

$$\frac{\partial\lambda(t)}{\partial t} \approx \Phi(\mathrm{r})^{-1}\left\{\frac{1}{2}\sigma^2\left[(1 - \sqrt{\frac{2}{\pi}}\tilde{C}\left(\sigma S\left|\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t)\right|\sqrt{\Delta t}\right)\frac{sgn\left(\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t)\right)}{\sigma\sqrt{\Delta t}}\right]S^2\frac{\partial^2}{\partial^2 S}\lambda(t) +\right.$$

$$\left. + rS\frac{\partial}{\partial S}\lambda(t) - r\lambda(t)\right\} \tag{23}$$

Similarly, replacing the approach to $V(S,t)$ given by Equation (17) added to the volatility model given by Equation (16), one can write equation (21) as:

$$\frac{\partial\lambda(t)}{\partial t}\Phi(\mathrm{r}) \approx \frac{1}{2}\sigma^2\left[\left(1+e^{r(T-t)}a^2S^2\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t)\right)\right]S^2\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t)++rS\frac{\partial\Phi(\mathrm{r})}{\partial S}\lambda(t)- -r\Phi(\mathrm{r})\lambda(t) \tag{24}$$

Multiplying both sides of the Equation (24) by $\Phi(\mathrm{r})^{-1}$ we have:

$$\frac{\partial\lambda(t)}{\partial t} \approx \Phi(\mathrm{r})^{-1}\left\{\frac{1}{2}\sigma^2\left[\left(1+e^{r(T-t)}a^2S^2\frac{\partial^2\Phi(\mathrm{r})}{\partial^2 S}\lambda(t)\right)\right]S^2\frac{\partial^2}{\partial^2 S}\lambda(t)+rS\frac{\partial}{\partial S}\lambda(t)-r\lambda(t)\right\} \tag{25}$$

Equations (23) and (25) generate a system of nonlinear equations, which can be solved to obtain the unknown coefficients $\boldsymbol{\lambda(t)}$ through the use of Runge-Kutta- Fehlberg's adaptive method (RKF). From this point on, they can be transformed into $\mathbf{V(S,t)}$ through Equation (17).

Runge-Kutta-Fehlberg adaptive method

According to Santos et al. (2009) most practical problems involve complex nonlinear problems, for which there are no analytical solutions. Therefore, these problems should be solved by means of numerical methods. Of particular importance for this work, Runge-Kutta-Fehlberg adaptive method (RKF) allows to obtain numerical approximations for the exact solution, within pre-user-specified errors, with automatically generated time intervals.

Consider an ordinary differential equation (ODE) in the following format:

$$\frac{\mathrm{dy}}{\mathrm{dt}} = f(t,y),\ \ y(t_0) = y_0 \tag{26}$$

where t is the independent variable and y is the dependent variable; subscript 0 refers to initial values.

The RKF adaptive method involves calculating two different order estimates. Thus, equation (26) can be solved by means of a fifth order method and error estimation, as the difference in the estimates of solutions obtained from fifth and fourth orders (Chapra & Canale, 2010):

Fifth Order Estimate:

$$y_{i+1} = y_i + \left(\frac{16}{135}k1 + \frac{6656}{12825}k3 + \frac{28561}{56430}k4 - \frac{9}{50}k5 + \frac{2}{55}k6\right) \tag{27}$$

Fourth Order Estimate:

$$y_{i+1} = y_i + \left(\frac{25}{216}k1 + \frac{1408}{2565}k3 + \frac{2197}{4104}k4 - \frac{k5}{5}\right) \quad (28)$$

in that,

$$k1 = f(t_i, y_i)$$

$$k2 = dt.f\left(t_i + \frac{1}{4}dt, y_i + \frac{1}{4}k1\right)$$

$$k3 = dt.f\left(t_i + \frac{3}{8}dt, y_i + \frac{3}{32}k1 + \frac{9}{32}k2\right)$$

$$k4 = dt.f\left(t_i + \frac{12}{13}dt, y_i + \frac{1932}{2197}k1 + \frac{7200}{2197}k2 + \frac{7296}{2197}k3\right) \quad (29)$$

$$k5 = dt.f\left(t_i + dt, y_i + \frac{439}{216}k1 + 8k2, \frac{3680}{513}k3 + \frac{845}{4104}k4\right)$$

$$k6 = dt.f\left(t_i + \frac{dt}{2}, y_i - \frac{8}{27}k1 + 2k2 - \frac{3544}{2565}k3 + \frac{1859}{4104}k4 - \frac{11}{40}k5\right)$$

where dt is the time step.

The difference between the values in the fourth and fifth order RKF methods for determining the relative error can be obtained by the following expression (Chapra & Canale, 2010):

$$erro = \frac{k1}{360} - \frac{128}{4275}k3 - \frac{2197}{75240}k4 - \frac{k5}{50} + \frac{2}{55}k6 \quad (30)$$

The time step is adjusted by referencing a predetermined truncation error value $\varepsilon\varepsilon$. If the difference in values obtained by the fourth and fifth order RKF methods are above the desired error parameter, the time step is consequently decreased by one factor F. Likewise, if the error value found is below the established parameter, the time step should be increased by the same factor. This process seeks to considerably improve computational processing time.

$$dt_{novo} = \frac{dt_{atual}}{F} \; se\ o\ erro\ \geq \varepsilon\varepsilon \quad (31)$$

$$dt_{novo} = dt_{atual}\ x\ F\ se\ o\ erro\ \leq \varepsilon\varepsilon$$

4. PRESENTATION AND DISCUSSION OF RESULTS

Numerical results for the nonlinear model proposed by Sevcovic and Zitnanská (2016)

This item presents the numerical results obtained for European Call options using the MQ RBF method for the nonlinear BS model with variable transaction costs as proposed by Sevcovic and Zitnanská (2016). Two curves obtained for the option price are also presented using the Leland (1985) Model of constant transaction cost, the first with lower volatility value(σ_{min}) and the second with higher volatility value (σ_{max}).

For simulation and analysis, similar data to those used in the work of Sevcovic and Zitnanská (2016) were used. Thus, the results obtained were based on the following input data:

- Initial Transaction Cost: $C_0 = 0.02$
- Final Transaction Cost: $\overline{C_0} = C_0 - \kappa(\xi_+ - \xi_-) = 5\, x\, 10^{-3}$
- Proportional Transaction Cost: $\kappa = 0.3$
- Exchange ranges of traded assets: $\xi_- = 0.05$ e $\xi_+ = 0.1$
- Asset exercise price: $K = 25$
- Volatility of the object asset: $\sigma = 0.3$
- Risk-neutral interest rate: $r = 0.011$
- Deadline for expiration of the option: $T = 1$
- Interval between two consecutive rearrangements in the portfolio: $\Delta t = \frac{1}{261}$

In order to compare the results obtained in this study, the following values were taken as a reference according to the results obtained by Sevcovic and Zitnanská (2016) for the nonlinear model.

- Reference solution for nonlinear model in $S = 20$: $V_{NLS}(20.0) = 0.127$
- Reference solution for nonlinear model in $\mathrm{S} = 23$: $V_{NLS}(23.0) = 0.844$
- Reference solution for nonlinear model in $\mathrm{S} = 25$: $V_{NLS}(25.0) = 1.748$
- Reference solution for nonlinear model in $\mathrm{S} = 28$: $V_{NLS}(28.0) = 3.695$
- Reference solution for nonlinear model in$S = 20$: $V_{NLS}(30.0) = 5.321$

For two linear models with volatility σ_{min} and σ_{max} the results for the option price (V_{min} and V_{max}) compared to the analytical solution. The analytical solution for the linear BS equation can be found in (During, 2005).

After tests performed with the Sevcovic and Zitnanská (2016) model the best results were obtained with the following simulation parameters to determine the value of the option numerically: initial dt = 1x10^{-3}, $\varepsilon\varepsilon$ = 1x10^{-8}, form factor, c, of MQ RBF equal to 0.0009, number of points N = 216, ΔS1 = 0.28, ΔS2 = 0.28/1.1 for S ∈ [24,25], and maximum value of S (underlying asset) equal to 60. It is noted that to solve this model, the use of a greater number of points (ΔS2 = 0.28/1.1) in the range of S ∈ [24,25] since with this strategy, there was a decrease in numerical errors at the inflection point of the initial condition (pay off function) of the problems V_{min} e V_{max}.

Figure 1 shows the graph referring to the variable transaction cost function $\tilde{C}(\xi)$ depending on the volume of assets traded (ξ) obtained through equation (9). Note that as the number of assets traded increases there is a reduction in the transaction cost amount within a range of $C_0 = 0{,}020$ e $\bar{C}_0 = 0{,}005$ as proposed by Sevcovic and Zitnanská (2016).

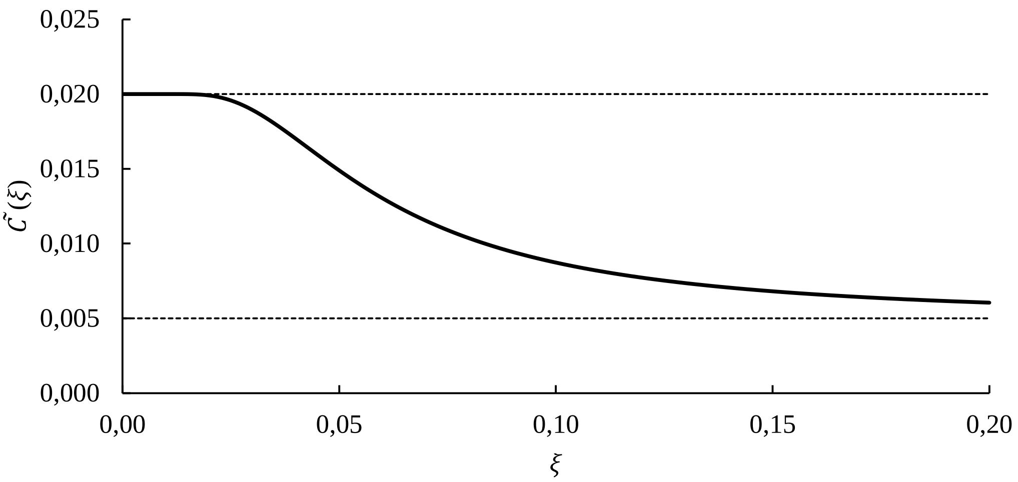

Figure 1. Variable transaction cost $\tilde{C}(\xi)$ depending on the volume of assets traded (ξ).

The numerical results found for the option price obtained through the nonlinear BS model with variable transaction cost, V_{NLS} , and through the linear model with constant transaction cost, V_{max} (with higher volatility), $\sigma_{max}^2 = \sigma^2\left(1 - \bar{C}_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right)$, and V_{min} (with lower volatility), $\sigma_{min}^2 = \sigma^2\left(1 - C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}\right)$, in $t = 0$, can be seen in Figure 2.

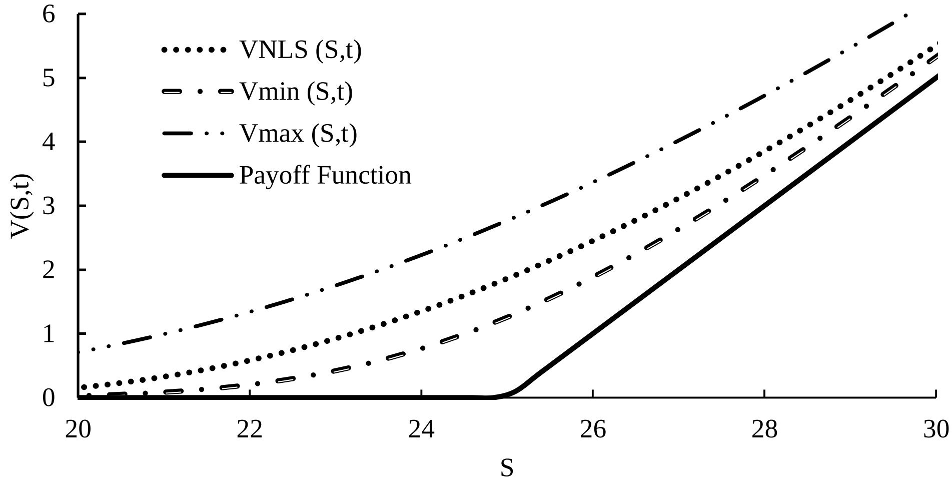

Figure 2 - Numerical results for the option price V_{NLS}, V_{max} e V_{min} in t = 0 compared to the payoff function.

You can see in Figure 2 that, in time t = 0, the value of the V_{NLS} approaches the curve of V_{min} with lower volatility parameter. As noted by Sevcovic and Zitnanská (2016), at the beginning of the contract, the investor does not need many rearrangements in his portfolio to hedge his position. In this way, the cost per transaction is equal to C_0. As time approaches the exercise in t = T, it becomes necessary to perform frequent rearrangements in the portfolio, increasing the traded assets. This increase carries a discounted transaction cost for the investor in the amount of $\bar{C}_0$.

The mean quadratic deviation of the numerical solution obtained with the nonlinear model (V_{NLS}) using the MQ RBF method, it was calculated by comparing with the five values presented in the work of Sevcovic and Zitnanská (2016), in S = 20, 23, 25, 28 and 30. The average quadratic error for V_{max} e V_{min} was calculated based on values obtained through analytical solution:

$$\varepsilon = Mean\ quadratic\ error\ or\ deviation\ of\ option\ price(\%) =$$

$$= \sqrt{\frac{1}{N}\ x\ \sum_0^N (Reference\ Value - Numerical\ Value)^2} \qquad (33)$$

being N the number of points considered for error calculation.

It can be observed, in Table 1, the errors calculated with and without the increase of points in the region of the inflection point of the initial condition of the problems V_{min}, V_{NLS} and V_{max}.

Table 1
Mean Quadratic Deviation / error calculated for numerical values of V_{NLS}, V_{max} and V_{min} with and without increasing points in the inflection region of the initial condition.

ΔS	V_{min} Mean Quadratic Error (ε)	V_{NLS} Mean Quadratic Deviation (ε)	V_{max} Mean Quadratic Error (ε)
ΔS = 0.28	5.28×10^{-4}	0.125	3.55×10^{-4}
ΔS1 = 0.28 **ΔS2=0.28/1.1**	4.41×10^{-4}	0.125	3.21×10^{-4}

According to the results presented in Table 1, there is an excellent approximation of the value of the option obtained by the MQ RBF method for the linear model, since the mean quadratic error of V_{min}e V_{max} was of 4.41 x 10^{-4} e 3.21 x 10^{-4} respectively. It is also possible to observe that with the increase in the number of points in the region of inflection point S [24,25] there was a decrease in the error calculated for V_{min} and V_{max} (reduction of 16.5% and 9.6% respectively), however, in the case of V_{NLS} there was no change in deviation.

The mean quadratic deviation obtained for the nonlinear BS equation with variable transaction costs compared to the result found in the work of Sevcovic and Zitnanská (2016) was 0.125. It is observed in Table 2 the effect of the variation on the number of N points in the mean quadratic deviation when using the MQ RBF method.

Table 2
Effect of variation on the number of N points in the Mean Quadratic Deviation (ε) when using the MQ RBF method.

Value of S	Option Value obtained in the work of Sevcovic and Zitnanská (2016)	Option Value V_{NLS} with N = 216	Option Value V_{NLS} with N = 40	Option Value V_{NLS} with N = 30	Option Value V_{NLS} with N = 24
20	0.127	0.154	0.136	0.128	0.113
23	0.844	0.923	0.886	0.85	0.791
25	1.748	1.861	1.818	1.799	1.683
27	3.695	3.852	3.826	3.811	3.812
30	5.321	5.505	5.495	5.481	5.439
Mean Quadratic Deviation (ε)	**0**	**0.125**	**0.104**	**0.091**	**0.083**

There is a greater approximation of the results obtained in this work in relation to the results of the work presented by Sevcovic and Zitnanská (2016) when there is a reduction in the number of points in the MQ RBF method. However, it is worth noting that, as the nonlinear BS model with variable transaction costs does not have an analytical solution, the mean quadratic deviation was calculated based on only five reference values – values presented in the work of Sevcovic and Zitnanská (2016).

Numerical Results for the Nonlinear Identity Model proposed by Barles and Soner (1998)

This item presents the numerical results obtained for the nonlinear BS model with volatility model Identity proposed by Barles and Sonner (1998) through the MQ RBF. The results were compared to those obtained in the work of Ankudinova (2008). A linear model with constant volatility (σ) is also presented, enabling a better visualization of the difference found in the option price when considering in the calculation the modified volatility function.

For simulation and analysis, similar data used in the work of Ankudinova (2008) were used. Thus, the results obtained were based on the following input data:

- Risk-free parameter: $a = 0{,}02$
- Asset exercise price: $K = 100$
- Volatility of the object asset: $\sigma = 0{,}2$
- Risk-neutral interest rate: $r = 0{,}1$
- Deadline for expiration of the option: $T = 1$

In order to compare the results obtained in this work with the results obtained by Ankudinova (2008), the following value was taken as reference:

- Reference solution for the nonlinear BS model in $S = 95$ $V_{NLBS}(95, T) = 10$

For the linear BS model with constant volatility the results for the option price (V_L) compared to the analytical solution. After tests with the model the best results were obtained with the following simulation parameters to determine the value of the option numerically: initial dt = $1x10^{-3}$, $\varepsilon\varepsilon$ = $1x10^{-8}$, form factor, c, of MQ RBF equal to 8.2, number of points N = 119, ΔS = 1.7, and maximum value of S (underlying asset) equal to 200. The use of a larger number of points has also been tested (ΔS2 = 1.7/1.1) in the range of S $\in$ [99,101] to check whether there would be a minimization of numerical errors at the inflection point of the initial condition (payoff function) of the problems V_{NLBS} e V_L. The numerical results found for the option price from the nonlinear BS model V_{NLBS} associated with the Identity volatility model proposed by Barles and Soner (1998) in $t = 0$ can be seen in Figure 3. The price of the option obtained from the solution of the linear equation of BS with constant volatility ($\sigma = 0{,}2$) is represented by V_L. A small difference in the price values of the option obtained by the two models is noted due to the effect of modified volatility incorporated in the nonlinear model. The payment function can also be seen in Figure 3.

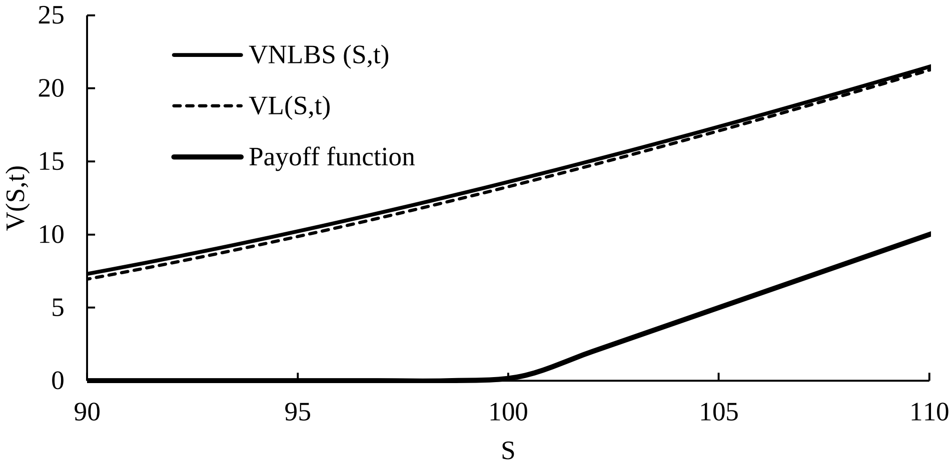

Figure 3 - Numerical results for the option price V_{NLBS} e V_L in $t = 0$ compared to the payment function.

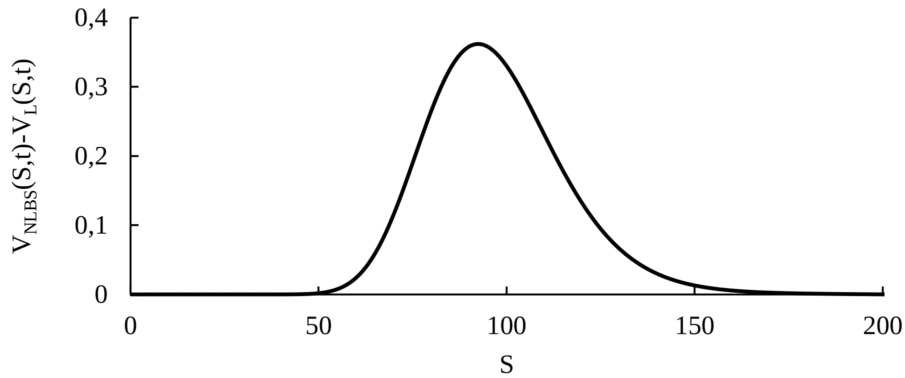

Figura 4 – Difference between option prices V_{NLBS} and V_L depending on the value of the underlying asset (S).

Because it was only presented in the work of Ankudinova (2008) the result of the option price for the nonlinear model Identity V_{NLBS} at one point (S=95), it was only possible to compare the results obtained in this study for this point. Thus, we chose to calculate the relative numerical deviation for the result found as follows:

$$\varepsilon_{relative} = Relative\ numerical\ deviation\ for\ option\ price(\%) =$$

$$= \frac{Numerical\ Value\ of\ the\ option - Value\ of\ the\ reference\ solution}{Value\ of\ the\ reference\ solution} x100\% \qquad (34)$$

Table 3 shows the relative deviation referring to the numerical value of V_{NLBS} and the mean quadratic error for the values of V_L with and without increasing points in the region of the inflection point of the initial condition.

Table 3

Deviation / error calculated for the numerical values of V_{NLBS} and V_L with and without increasing points in the inflection region of the initial condition.

ΔS	V_L Mean Quadratic Error (ε)	V_{NLBS} Relative Numerical Deviation ($\varepsilon_{relative}$)
ΔS = 1.7	6.36x10^{-4}	0.022
ΔS1 = 1.7 **ΔS2 = 1.7/1.1**	6.62x10^{-4}	0.022

According to the results presented in Table 3, there is an excellent approximation of the value of the option obtained by the MQ RBF method for the linear model, since the mean quadratic error was 6.36x10^{-4}. It is also possible to observe that with the increase of points in the region of the inflection point S ∈ [99,101] there is no considerable change in the error calculated for V_L (increase of 4.1% in error) and no change in the relative deviation of V_{NLBS}.

The relative deviation obtained for the nonlinear BS equation with Identity volatility model compared with the result found in the Ankudinova (2008) work was 0.022. It is observed in Table 4 the effect of the variation in the number of points N in the relative deviation to the use of the Method of MQ RBF.

Table 4

The effect of the variation in the number of points N in the relative deviation ($\varepsilon_{relative}$) to the use of the Method of MQ RBF.

Value of S	Option Value obtained in work of Ankudinova (2008)	Option Value V_{NLBS} with N = 119	Option Value V_{NLBS} with N = 40	Option Value V_{NLBS} with N = 20	Option Value V_{NLBS} with N = 13
95	10	10.221	10.214	10.047	10.08
Relative Deviation ($\varepsilon_{relative}$)	**-**	**0.022**	**0.021**	**4.70x10^{-3}**	**8.01x10^{-3}**

A greater approximation of the results obtained in this work with the results of the work presented by Ankudinova (2008) is observed when there is a decrease in the number of points used in the MQ RBF method. It is noteworthy that since the nonlinear BS model with identity volatility model does not have an analytical solution, the relative deviation was calculated based on only one reference value – a value obtained in the work presented by Ankudinova (2008).

Analysis of the efficiency of the adaptive method for the solution of non-linear and linear Black-Scholes

Table 5 and 6 show the total number of iterations (time steps) required to achieve the final simulation time (T=1), with and without the use of an RKF temporal adaptive method for nonlinear BS configurations for European call options presented in the works of Sevcovic and Zitnanská (2016) and Ankudinova (2008), respectively. Additionally, the deviations obtained through the proposed methods and the results presented by the authors mentioned above are shown, considering the initial time step (initial Δt) equal to 0.001 and truncation error ($\varepsilon\varepsilon$), set to adaptive method, equal to $1\text{x}10^{-8}$.

Table 5

Total number of iterations required to achieve the final simulation time (T=1) and mean quadratic deviation with and without the use of adaptive method - Sevcovic and Zitnanská model (2016).

Factor F (Equation 32)	Number of Iterations		Quadratic Mean Deviation	
	Non-Adaptative Method	Adaptative Method	Non-Adaptative Method	Adaptative Method
1.0001	1000	954	0.125	0.125
1.0003		875		0.125
1.0005		812		0.126

From the results presented in Table 5, it is noted that using the adaptive method RKF with 812 iterations is obtained an average quadratic deviation close to the method without adaptability in time with 1000 iterations. Therefore, there is a gain in the number of iterations without considerable loss of accuracy in the results when using the numerical method with temporal adaptability. The reduction in the number of iterations consequently lead to the decrease in computational time for solution of the nonlinear model, which is extremely important in practice.

Table 5 also shows that there is no variation in deviation with the application of the adaptive method in time RKF applied to the nonlinear BS model proposed by Sevcovic and Zitnanská (2016), when the F factor was varied to update the time step between 1.0003 and 1.

Table 6

Total number of iterations required to achieve the final simulation time (T=1) and relative deviation with and without the use of adaptive method - Barles and Soner Model (1998) presented in the work of Ankudinova (2008).

Factor F (Equation 32)	Number of Iterations		Relative Deviation	
	Non-Adaptative Method	Adaptative Method	Non-Adaptative Method	Adaptative Method
1.0001	1000	954	0.022	0.023
1.0002		912		0.022
1.0004		842		0.023

From the results presented in Table 6 it is possible to observe that the use of the adaptive method with 842 iterations obtained a relative deviation close to the method without adaptability in time with 1000 iterations. Therefore, there is a gain in the number of iterations without considerable loss of accuracy in the results when using the numerical method with temporal adaptability. Again, it is noteworthy that the reduction in the number of iterations consequently lead to the decrease in computational time for the solution of the nonlinear model.

Table 6 also shows that good results were achieved with the adaptive method in time applied to the nonlinear BS model proposed by Barles and Soner (1998) and presented in the Ankudinova (2008) work, when the F factor was maintained to update the time step lower than 1.0004. For F values above 1.0004 the results diverged.

Table 7 shows the total number of iterations (time steps) required to achieve the final simulation time (T=1) with and without the use of an RKF temporal adaptive method for solution of the linear BS model for European purchasing options V_{min}, V_{max} and V_L, respectively. Additionally, the errors obtained through the proposed methods and the analytical solution of the linear BS model are shown considering the initial time step (Δt initial) equal to 0.001 and truncation error ($\varepsilon\varepsilon$), set to the adaptive method, equal to $1x10^{-8}$.

Table 7

Total number of iterations required to achieve the final simulation time (T=1) and mean quadratic error with and without the use of adaptive method to $V_{min}, V_{max}, e\ V_L$.

Linear BS models	Factor F (Equation 32)	Number of Iterations		Quadratic Mean Deviation	
		Non-Adaptative Method	Adaptative Method	Non-Adaptative Method	Adaptative Method
Vmin	1.0001	1000	954	4.41×10^{-4}	5.08×10^{-4}
	1.0003		875		4.50×10^{-4}
	1.0005		812		5.68×10^{-4}
Vmax	1.0001	1000	954	3.21×10^{-4}	7.70×10^{-4}
	1.0003		875		4.60×10^{-4}
	1.0005		812		1.02×10^{-3}
V_L	1.0001	1000	954	6.36×10^{-4}	6.39×10^{-3}
	1.0002		912		2.42×10^{-3}
	1.0004		842		6.28×10^{-3}

According to the results of Table 7, it is observed that for V_{min} there was an absolute variation between the highest and lowest error obtained from 1.27 $\times 10^{-4}$ with the number of iterations in time ranging from 812 to 1000 and factor F between 1 and 1.0005. For V_{max}, the absolute variation of the error was 6.99 x 10^{-4} with the number of iterations in time ranging from 812 to 1000 and factor F between 1 and 1.0005, as it was found for the V_{min}. Finally, for V_L, the absolute variation of the error was 5.75×10^{-3} with the number of iterations in time ranging from 842 to 1000 and factor F between 1 and 1.0004. It is noteworthy that for F values above the upper limit of the described intervals there was a considerable increase in the numerical errors calculated. It is concluded that, through the adaptive method, good results are obtained in the linear model with reduction in the number of iterations and, consequently, in computational time.

Figure 5 shows the variation of the time step at each iteration due to the implementation of temporal adaptability (RKF method) in the solution of nonlinear and linear BS problems. Figure 5 shows the results for the smallest number of iterations obtained in the solution of problems, as shown in Tables 5, 6 and 7.

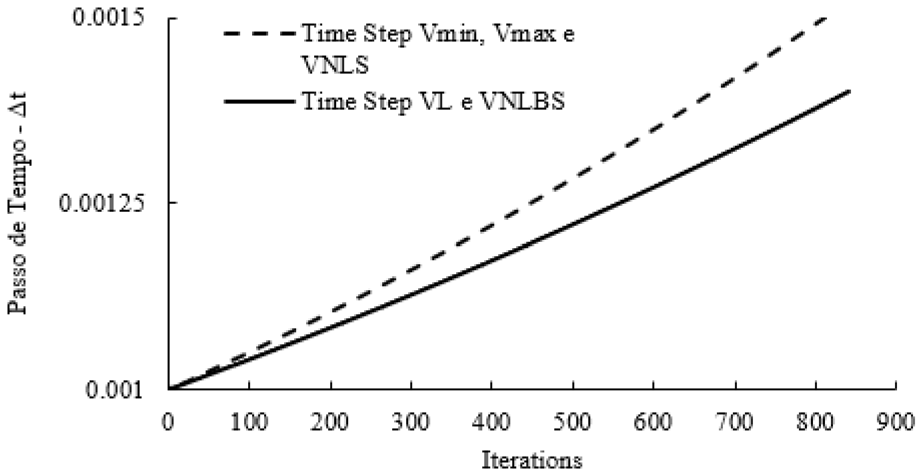

Figure 5 - Time step variation at each iteration due to the implementation of temporal adaptability.

It is observed that the time step is corrected to a higher value with each iteration, since the difference of the results obtained through Runge-Kutta- Fehlberg method of fourth and fifth order (Equation 30) for V_{min}, V_{max}, V_{NLS}, V_L and V_{NLBS} are always smaller than the truncation error $\varepsilon\varepsilon$ = 1x10^{-8} pre-established. The increase in the time step makes it possible to obtain the final solution of the problem with a smaller number of iterations, which generates, consequently, a gain in computational time that is extremely important for those who work in the financial market.

5. CONCLUSIONS

The main conclusions of this work are:

- Numerical results allow us to affirm that the methods of MQ RBF, adaptive and non-adaptive in time, led to accurate and fast results, when applied to linear problems, as well as to problems of the nonlinear type with modified volatility. In the nonlinear models analyzed in this work - Sevcovic and Zitnanská (2016) and Ankudinova (2008) - there was a reduction in the deviations calculated when a lower number of points was used in the numerical solution obtained by the MQ RBF method.
- As for the increase in the number of points in the discontinuity region of the payment function, there was an improvement in the accuracy of the results calculated for V_{min} and V_{max}. In the other cases, there was no significant improvement in the results that justified its use.

- Regarding to the increase in the Values of Factor F of temporal adaptability used in the adaptive method in time for nonlinear models, there was a reduction in the number of iterations and, consequently, in computational time, keeping the numerical deviations practically unchanged (variation of 0,001 in deviation to V_{NLS} e V_{NLBS}). In the case of linear models, with the increase of factor F in the adaptive method, there was a reduction in the number of iteration and, consequently, in computational time, keeping the numerical errors in the order of $1x10^{-3}$ and $1x10^{-4}$.
- For a truncation error of $\varepsilon\varepsilon$ = $1x10^{-8}$, there was a greater variation in the time step through the adaptive method for the solution of V_{min}, V_{max} and V_{NLS} if compared to the solution of V_{NLBS}, V_L.

It is noteworthy that there is no analytical solution for the nonlinear models presented in this work, and, therefore, the deviation were calculated based on only one reference value - obtained in the work of Ankudinova (2008) for the nonlinear model of Barles and Soner (1998) - and based on five values presented in the work of Sevcovic and Zitnanská (2016) for the nonlinear model with variable transaction cost.

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